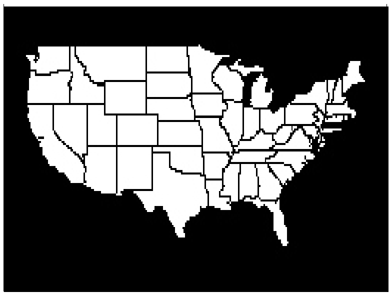


Set Theory
Lesson 9
Boolean Algebra of Sets

9.1 - What is Boolean algebra of Sets?

Boolean algebra is a powerful method for solving many problems in probability theory. Since we are studying sample spaces, we will define a Boolean algebra of sets in the following way: Given a family of sets in a sample space **A**, **B**, **C**, **D**, **E**..., we define two operations union $\cup$ and intersection $\cap$. Unless otherwise indicated with parentheses, intersections have priority over unions. For these operations, the following laws hold:

1. **Closure Laws:**

$\mathbf{A}\cup\mathbf{B} = \mathbf{E}$

$\mathbf{A}\cap\mathbf{B} = \mathbf{F}$,where **E** and **F** are unique.

2. **Commutative Laws:**

$\mathbf{A}\cup\mathbf{B} = \mathbf{B}\cup\mathbf{A}$

$\mathbf{A}\cap\mathbf{B} = \mathbf{B}\cap\mathbf{A}$

3. **Associative Laws:**

$(\mathbf{A}\cup\mathbf{B})\cup\mathbf{C} = \mathbf{A}\cup(\mathbf{B}\cup\mathbf{C})$

$(\mathbf{A}\cap\mathbf{B})\cap\mathbf{C} = \mathbf{A}\cap(\mathbf{B}\cap\mathbf{C})$

4. **Distributive Laws:**

$\mathbf{A}\cup(\mathbf{B}\cap\mathbf{C}) = (\mathbf{A}\cup\mathbf{B})\cap(\mathbf{A}\cup\mathbf{C})$

$\mathbf{A}\cap(\mathbf{B}\cup\mathbf{C}) = (\mathbf{A}\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{C})$

5. **Identity Laws:**

For all sets **A** there exists sets denoted by ϕ and **S** where

$\mathbf{A}\cup\phi = \mathbf{A}$

$\mathbf{A}\cap\mathbf{S} = \mathbf{A}$

6**. Complement Laws:**

For each set **A** there exists a set **A**′ where

A∪**A**′ = **S**.

A∩**A**′ = ϕ.

For this algebra of sets, the set **S** is called the sample space and ϕ the null set.

From these laws, the following properties can be proven:

Uniqueness of the complement:

1. If **B**∩**A** = ϕ and **B**∪**A** = **S** then **B** = **A**′.

DeMorgan Laws:

2. (**A**∪**B**)′ = **A**′∩**B**′

and

(**A**∩**B**)′ = **A**′∪**B**′

3**. A**∪**A = A**

A∩**A = A**

4. **A**∪**S = S**

A∩ϕ = ϕ

5**. A**′′= **A**

6**. S**′ = ϕ and ϕ′ = S

Definition of subsets

A is said to be a subset of **B**, denoted by **A** ⊆ **B**, if **A**∩**B**′ = ϕ.

A is said to be a proper subset of **B** if **A** is a subset of **B** but **B** is a not a subset of **A**.

Definition of equality of sets: A = **B** if **A** is subset of **B** and **B** is a subset of **A**.

From the definition of subsets, we have the following properties which can be proven:

1. **Reflexive Law**: **A** ⊆ **A** for all sets **A**.

2. **Anti-symmetric Law**: If **A** ⊆ **B** and **B** ⊆ **A** then **A** = **B**.

3. **Transitive Law**: If **A** ⊆ **B** and **B** ⊆ **C** then **A** ⊆ **C**.

4. **Complement Law**: If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{B}' \subseteq \mathbf{A}'$

5. **Universal Set S**: All the sets are subsets of **S**.

6. **Uniqueness law**:

If $\mathbf{A} \cap \mathbf{B} = \phi$ and $\mathbf{A} \cup \mathbf{B} = \mathbf{S}$ then $\mathbf{B} = \mathbf{A}'$.

7. **Subdivision Law:**

$\mathbf{A} \cup \mathbf{B} = (\mathbf{A}' \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{B}') \cup (\mathbf{A} \cap \mathbf{B})$

8. If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{B} = \mathbf{B} \cup \mathbf{A}$

9. If $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{A} = \mathbf{B} \cap \mathbf{A}$

Applications

The sets in our boolean algebra are events in sample spaces that are generated by well defined experiments. To translate the events into boolean expressions, we use the following: the words **and**, **or** and **not** are used to express events in conjunction with intersections, union and complements. The following table shows the relationship of these expressions to the events:

Set	Expression	Key Word
$\mathbf{A}'$	The event **A** does not occur.	**NOT**
$\mathbf{A} \cap \mathbf{B}$	The events **A** and **B** both occur.	**AND**
$\mathbf{A} \cup \mathbf{B}$	The events **A** or **B** or both occur.	**OR**

9.1 - Example 1: Assume two cards are drawn from an ordinary deck of cards. Assume $\mathbf{K}_1$ is the event that the first card drawn is a king and $\mathbf{K}_2$ is the event that the second card drawn is a king. Write out the event **E** that

(a). both cards drawn are kings.

(b). only one card is a king.

(c). no card drawn is a king.

(d). at least one card drawn is a king.

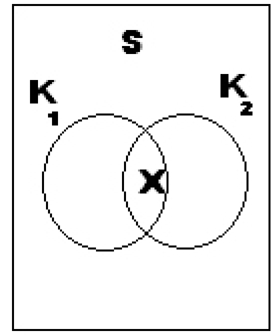
(e). Simplify $\mathbf{K_2 \cap (K_1 \cup K_2 ')}$.

(f). Simplify $\mathbf{K_2 \cup (K_1 \cap K_2 ')}$.

Solutions:

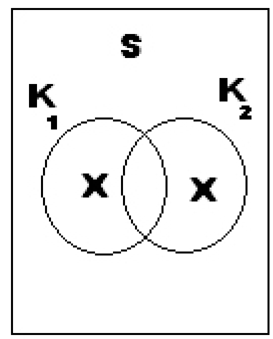
➤ **(a)**.
The event **E** can be stated as follows: the first card and the second card is a king. Since the word and is associated with the intersection $\cap$, we can write $\mathbf{E = K_1 \cap K_2}$.

a.



➤ **(b)**.
Since only one card is a king, we use the events $\mathbf{K_1}$,$\mathbf{K_2}$, $\mathbf{K_1 '}$, $\mathbf{K_2 '}$. The event, "only one card is a king", means the following:

b.



1. If a king occurs on the first drawing then a king cannot occur on the second drawing:$\mathbf{(K_1 \cap K_2 ')}$.

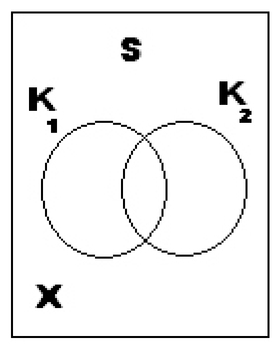
or

2. If a king occurs on the second drawing then a king cannot occur on the first drawing:$\mathbf{(K_1 ' \cap K_2)}$.

3. We can write this event as $\mathbf{E = (K_1 \cap K_2 ') \cup (K_1 ' \cap K_2)}$.

➤ **(c)**.
The event , "no card drawn is a king", means that the first **and** the second card are not kings. Therefore, we can write this event as $\mathbf{E = K_1 ' \cap K_2 ' = (K_1 \cup K_2)'}$.

c.



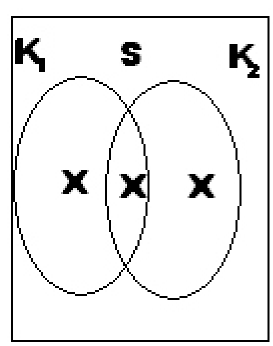
➤ **(d)**.
The event, "at least one card drawn is a king, means that one **or** two cards drawn are kings. This can happen in the following ways:

1. The first card drawn is a king **and** second card is not a king: $\mathbf{K_1 \cap K_2 '}$.

or

2. The second card drawn is a king **and** first card is not a king: $\mathbf{K_1 ' \cap K_2}$.

d.



or

3. Both cards drawn are kings: $\mathbf{K_1 \cap K_2}$.

4. This can be written as $\mathbf{E = (K_1 \cap K_2 ') \cup (K_1 ' \cap K_2) \cup (K_1 \cap K_2) = K_1 \cup K_2}$.

➤ **(e)**.
From the distributive law , we can write

$\mathbf{K_2 \cap (K_1 \cup K_2') = (K_2 \cap K_1) \cup (K_2 \cap K_2') = (K_1 \cap K_2) \cup \phi = K_1 \cap K_2}$.

➤ **(f)**.
From the distributive law , we can write

$\mathbf{K_2 \cup (K_1 \cap K_2') = (K_2 \cup K_1) \cap (K_2 \cup K_2') = (K_1 \cup K_2) \cap S = K_1 \cup K_2}$.

9.1 - Example 2: An urn contains several red, white and blue marbles. Three marbles are selected at random, one at a time. Let $\mathbf{R_i}$, $\mathbf{W_i}$, $\mathbf{B_i}$ (i = 1, 2, 3) represent the results of the i th color marble being selected. Find the events **E:**

(a). two marbles are blue and one is red.

(b). Exactly two marbles are blue.

(c). No red marble is selected.

(d). the first two marbles are red and the third is white.

(e). exactly one of each appears.

Solutions:

➤ **(a)**.
The event **E** can happen in the following ways:

1. first is blue **and** second is blue **and** third is red: $\mathbf{B_1 \cap B_2 \cap R_3}$.

or

2. first is blue **and** second is red **and** third is blue: $\mathbf{B_1 \cap R_2 \cap B_3}$.

or

3. first is red **and** second is blue **and** third is blue:$\mathbf{R_1 \cap B_2 \cap B_3}$.

4. Therefore, $\mathbf{E = (B_1 \cap B_2 \cap R_3) \cup (B_1 \cap R_2 \cap B_3) \cup (R_1 \cap B_2 \cap B_3)}$.

➤ **(b)**.
The event **E** can happen in the following ways:

1. The first and second marbles are blue and the third is not blue: $\mathbf{B_1 \cap B_2 \cap B_3'}$.

or

2. The first and third marbles are blue and the second is not blue: $\mathbf{B_1 \cap B_2' \cap B_3}$.

or

3. The Second and third marbles are blue and the first is not blue: $\mathbf{B_1}'\cap\mathbf{B_2}\cap\mathbf{B_3}$.

4. Therefore, $\mathbf{E} = (\mathbf{B_1}\cap\mathbf{B_2}\cap\mathbf{B_3}')\cup(\mathbf{B_1}\cap\mathbf{B_2}'\cap\mathbf{B_3})\cup(\mathbf{B_1}'\cap\mathbf{B_2}\cap\mathbf{B_3})$.

➤ **(c)**.
This event can happen in the following way: no red marble on the first selection ($\mathbf{R_1}'$), **and** no red marble on the second selection ($\mathbf{R_2}'$), **and** no red marble on the third selection $\mathbf{R_3}'$.Joining these events with intersections, we have $\mathbf{E} = \mathbf{R_1}'\cap\mathbf{R_2}'\cap\mathbf{R_3}'$.

➤ **(d)**.
Since the first two selected are red and the last drawn is white, we can write the event as $\mathbf{E} = \mathbf{R_1}\cap\mathbf{R_2}\cap\mathbf{W_3}$.

➤ **(e)**.
There are six ways this can happen:

1. The first selection is red, the second selection is blue and the third selection is white: $\mathbf{R_1}\cap\mathbf{B_2}\cap\mathbf{W_3}$.

or

2. The first selection is red, the second selection is white and the third selection is blue: $\mathbf{R_1}\cap\mathbf{W_2}\cap\mathbf{B_3}$.

or

3. The first selection is white, the second selection is blue and the third selection is red: $\mathbf{W_1}\cap\mathbf{B_2}\cap\mathbf{R_3}$.

or

4. The first selection is white, the second selection is red and the third selection is blue: $\mathbf{W_1}\cap\mathbf{R_2}\cap\mathbf{B_3}$.
or

5. The first selection is blue, the second selection is white and the third selection is red: $\mathbf{B_1}\cap\mathbf{W_2}\cap\mathbf{R_3}$.

or

6. The first selection is blue, the second selection is red and the third selection is white: $\mathbf{B_1}\cap\mathbf{R_2}\cap\mathbf{W_3}$.

7. Therefore,

$$\mathbf{E} = (\mathbf{R_1}\cap\mathbf{B_2}\cap\mathbf{W_3})\cup(\mathbf{R_1}\cap\mathbf{W_2}\cap\mathbf{B_3})\cup(\mathbf{W_1}\cap\mathbf{B_2}\cap\mathbf{R_3})\cup(\mathbf{W_1}\cap\mathbf{R_2}\cap\mathbf{B_3})\cup(\mathbf{B_1}\cap\mathbf{W_2}\cap\mathbf{R_3})\cup(\mathbf{B_1}\cap\mathbf{R_2}\cap\mathbf{W_3}).$$

9.1 - Example 3: Ms. Jones receives at least 10 calls a day.

Let **A** be the event she receives at most 15 calls a day;
Let **B** be the event she receives at least 12 calls a day;
Let **C** be the event she receives between 15 and 20 calls a day.

Using these events, find the following events **E**:

(a). She receives at least 16 calls a day.

(b). She receives 10 or 11 calls a day.

(c). She receives between 12 and 15 calls a day (inclusive).

(d). She receives exactly 15 calls a day.

Solutions:

➤ **(a)**.
The sample space **S** = {10, 11, 12,...}.

Step 1: **A** = {10, 11, 12, 13, 14, 15} and

Step 2: **A**′ = {16, 17, 18,...}.

Step 3: Therefore, **E** = **A**′.

➤ **(b)**.
Step 1: **B** = {12, 13, 14,...}

Step 2: **B**′ = {10, 11}

Therefore, **E** = **B**′.

➤ **(c)**.
Step 1: **A** = {10, 11, 12, 13, 14, 15}

Step 2: **B** = {12, 13, 14, 15, 16,...}

Step 3: **A**∩**B** = {12, 13, 14, 15}

Step 4: Therefore, **E** = **A**∩**B**.

➤ **(d)**.
Step 1: **A =** {10, 11, 12, 13, 14, 15}

Step 2: **C** = {15, 16, 17, 18, 19, 20}

Step 3: **A**∩**C** = {15}

Step 4: Therefore, **E** = **A**∩**C**.

9.1 - Example 4: Two urns each contain red and white marbles. An urn is selected at random and a red marble is selected. Let $\mathbf{U_A}$ represent the event that urn A was selected and **R** the event that a red marble is selected. Write **R** as an expression of **R**, $\mathbf{U_A}$ and $\mathbf{U_A}'$.

Solution:

The event that a red marble is selected depends on which urn was selected.

1. We can write the event that a red marble is selected came from urn A as $\mathbf{R \cap U_A}$.

or

2. We can write the event a red marble is selected from urn B as $\mathbf{R \cap U_A}'$. Connecting these two event, $\mathbf{R} = (\mathbf{R \cap U_A}) \cup (\mathbf{R \cap U_A}')$.

9.1 - Example 5: Two cards are drawn from an ordinary deck of cards. Let $\mathbf{H_1}$ be the event that a heart was drawn on the first drawing, $\mathbf{H_2}$ be the event that a heart was drawn on the second drawing. Write $\mathbf{H_2}$ as an expression of $\mathbf{H_1}$ and $\mathbf{H_2}$.

Solution:

1. A heart is drawn on the first **and** second drawing: $\mathbf{H_2} = \mathbf{H_1 \cap H_2}$.

or

2. A heart is drawn on the second **but not** on the first drawing: $\mathbf{H_2} = \mathbf{H_1' \cap H_2}$.

3. Connecting these two event gives $\mathbf{H_2} = (\mathbf{H_1 \cap H_2}) \cup (\mathbf{H_1' \cap H_2})$.

Solved Problems

9.1 - Solved Problem 1: Assume three cards are drawn from an ordinary deck of cards. If $\mathbf{K_i}$ (i = 1, 2, 3) are the events that the i th card drawn is a king, write out the event **E** that

(a). all three cards drawn are kings.

(b). only one card is a king.

(c). no card drawn is a king.

(d). at least one card drawn is a king.

(e). Simplify $\mathbf{K_1 \cap [(K_1 \cap K_2) \cup (K_2 \cap K_1')]}$.

(f). Simplify $\mathbf{K_1 \cup [(K_1 \cup K_2) \cap (K_2 \cup K_1')]}$.

Solutions:

➤ **(a)**.

The event **E** can be stated as follows: the first card **and** the second card **and** the third card is a king. Since the word **and** is associated with the intersection ∩, we can write $\mathbf{E = K_1 \cap K_2 \cap K_3}$.

➤ **(b)**.

The event, "only one card is a king", can occur in the following ways:

1. king is drawn first followed by two cards that are not kings: $\mathbf{K_1 \cap K_2' \cap K_3'}$.

or

2. the first card drawn is not a king, the second card drawn is a king and the third card drawn is not a king: $\mathbf{K_1' \cap K_2 \cap K_3'}$.

or

3. the first card drawn is not a king, the second card drawn is not a king and the third card drawn is a king: $\mathbf{K_1' \cap K_2' \cap K_3}$.

Therefore, $\mathbf{E} = (\mathbf{K_1 \cap K_2' \cap K'_3}) \cup (\mathbf{K_1' \cap K_2 \cap K_3'}) \cup (\mathbf{K_1' \cap K_2' \cap K_3})$.

➤ **(c)**.

The event, "no card drawn is a king", can occur in the following way: the first card drawn is not a king, the second card drawn is not a king and the third card drawn is not a king: $\mathbf{K_1' \cap K_2' \cap K_3'}$.

➤ **(d)**.

Step 1: E: the event that at least one card drawn is a king.

E′: the event that no card drawn is a king.

Step 2: From (c) we have $\mathbf{E'} = \mathbf{K_1' \cap K_2' \cap K_3'}$.

Step 3: $\mathbf{E} = \mathbf{E''} = (\mathbf{K_1' \cap K_2' \cap K_3'})' = \mathbf{K_1'' \cup K_2'' \cup K_3''} = \mathbf{K_1 \cup K_2 \cup K_3}$.

➤ **(e)**.

Using the distributive law,

$$\mathbf{K_1 \cap [(K_1 \cap K_2) \cup (K_2 \cap K_1')] = [K_1 \cap (K_1 \cap K_2)] \cup [K_1 \cap (K_2 \cap K_1')]. = (K_1 \cap K_2) \cup (\phi \cap K_2) = K_1 \cap K_2}.$$

➤ **(f)**.

Using the distributive

$$\mathbf{K_1 \cup [(K_1 \cup K_2) \cap (K_2 \cup K_1')] = [K_1 \cup K_1 \cup K_2)] \cap [K_1 \cup (K_2 \cup K_1')] = (K_1 \cup K_2) \cap (S \cup K_2) = (K_1 \cup K_2) \cap S = K_1 \cup K_2}.$$

9.1 - Solved Problem 2: Mrs. Jones teaches third grade. In a recent survey from her students, she found that the only flavors of ice cream the students like are strawberry, chocolate, and raspberry. Three children are selected one at a time. Find the event **E**.

(a). two students like chocolate and one likes raspberry.

(b). Exactly two students like raspberry.

(c). No student likes strawberry.

(d). the first two students selected like strawberry and the third student likes chocolate.

(e). One student likes chocolate, one student likes strawberry and one student likes raspberry.

Solutions:

We use the following sets:

$\mathbf{R_i}$: the i th child selected likes raspberry.

$\mathbf{C_i}$ **:** the i th child selected likes chocolate.

$\mathbf{S_i}$: the i th child selected likes strawberry.

➤ **(a)**.
The event **E** can happen in the following ways:

1. first student likes chocolate **and** second likes chocolate **and** third likes raspberry: $\mathbf{C_1 \cap C_2 \cap R_3}$.

or

2. first student likes chocolate **and** second likes raspberry **and** third likes chocolate: $\mathbf{C_1 \cap R_2 \cap C_3}$.

or

3. first student likes raspberry **and** second likes chocolate **and** third likes chocolate: $\mathbf{R_1 \cap C_2 \cap C_3}$.

Therefore, $\mathbf{E} = (\mathbf{C_1 \cap C_2 \cap R_3}) \cup (\mathbf{C_1 \cap R_2 \cap C_3}) \cup (\mathbf{R_1 \cap C_2 \cap C_3})$.

➤ **(b)**.
The event **E** can happen in the following ways:

1. The first **and** second students like raspberry **and** the third student does not likes raspberry: $\mathbf{R_1 \cap R_2 \cap R_3}'$.

or

2. The first **and** third students like raspberry **and** the second student does not like raspberry: $\mathbf{R_1 \cap R_2' \cap R_3}$.

or

3. The second **and** third students like raspberry **and** the first student does not like raspberry: $\mathbf{R_1{}'\cap R_2\cap R_3}$.

Connecting these three expressions, we get $\mathbf{E} = (\mathbf{R_1\cap R_2\cap R_3{}'})\cup(\mathbf{R_1\cap R_2{}'\cap R_3})\cup(\mathbf{R_1{}'\cap R_2\cap R_3})$.

➤ **(c)**.
This event can happen in the following way: the first student does not like strawberry ($\mathbf{S_1{}'}$), **and** the second student does not like strawberry ($\mathbf{S_2{}'}$), **and** the third student does not like strawberry $\mathbf{S_3{}'}$.

Joining these events with intersections, we have $\mathbf{E = S_1{}'\cap S_2{}'\cap S_3{}'}$.

➤ **(d)**.
Since the first two students selected like strawberry and the last student likes chocolate, we can write the event as $\mathbf{E} = \mathbf{S_1\cap S_2\cap C_3}$.
➤ **(e)**.
There are six ways this can happen:

1. The first likes chocolate, the second student likes strawberry and the third student likes raspberry: $\mathbf{C_1\cap S_2\cap R_3}$.
or

2. The first likes chocolate, the second student likes raspberry and the third student likes strawberry: $\mathbf{C_1\cap R_2\cap S_3}$.
or

3. The first likes strawberry, the second student likes raspberry and the third student likes chocolate: $\mathbf{S_1\cap R_2\cap C_3}$.
or

4. The first likes strawberry, the second student likes chocolate and the third student likes raspberry: $\mathbf{S_1\cap C_2\cap R_3}$.
or

5. The first likes raspberry, the second student likes chocolate and the third student likes strawberry: $\mathbf{R_1\cap C_2\cap S_3}$.
or

6. The first likes raspberry, the second student likes strawberry and the third student likes raspberry: $\mathbf{R_1\cap S_2\cap C_3}$.
.
7. Therefore, $\mathbf{E} =(\mathbf{C_1\cap S_2\cap R_3})\cup(\mathbf{C_1\cap R_2\cap S_3})\cup(\mathbf{S_1\cap R_2\cap C_3})\cup(\mathbf{S_1\cap C_2\cap R_3})\cup(\mathbf{R_1\cap C_2\cap S_3})\cup(\mathbf{R_1\cap S_2\cap C_3})$.

9.1 - Solved Problem 3: Mr. Jones has at least three grandchildren.

let **A** be the event he has at most 10 grandchildren.

Let **B** be the event he has at least 5 grandchildren.

Let **C** be the event he has between 10 and 25 grandchildren.

Using these events, find the following events **E**:

(a). he has at least 11 grandchildren.

(b). he has 3 or 4 grandchildren.

(c). he has between 5 and 10 grandchildren.

(d). he has exactly 10 grandchildren.

Solutions:

➤ **(a)**.
The sample space **S** = {3, 4, 5, 6,...}.

Step 1: **A** = {3, 4,..., 10}

Step 2: **A**′ = {11, 12,...}

Therefore, **E** = **A**′.

➤ **(b)**.
Step 1: **B** = {5, 6, 7...}

Step 2: **B**′ = {3, 4}

Therefore, **E** = **B**′.

➤ **(c)**.
Step 1: **A** = {3, 4, 5,..., 10}

Step 2: **B** = {5, 6, 7,...}

Step 3: **A**∩**B** = {5, 6, 7, 8, 9, 10}

Step 4: Therefore, **E** = **A**∩**B**.

➤ **(d)**.
Step 1: **A** = {3, 4, 5,..., 10}

Step 2: **C** = {10, 11,..., 25}

Step 3: **A**∩**C** = {10}

Step 4: Therefore, **E** = **A**∩**C**.

9.1 - Solved Problem 4: Three urns each contain red marbles, white marbles and black marbles. A marble is selected from one of these urns. Let $\mathbf{U}_A$ represent the event that urn A was selected, $\mathbf{U}_B$ represent the event that urn B was selected and, $\mathbf{U}_C$ represent the event that urn C was selected. If **R** is the event that a red marble is selected, write **R** as an expression of **R** and $\mathbf{U}_A$, $\mathbf{U}_B$, $\mathbf{U}_C$.

Solution:

The event that a red marble is selected depends on which urn was selected.

1. We can write the event that a red marble is selected came from urn A as: $\mathbf{R \cap U_A}$.

or

2. We can write the event a red marble is selected from urn B as: $\mathbf{R \cap U_B}$.

or

3. We can write the event a red marble is selected from urn C as: $\mathbf{R \cap U_C}$.

Connecting these three event, $\mathbf{R} = (\mathbf{R \cap U_A}) \cup (\mathbf{R \cap U_B}) \cup (\mathbf{R \cap U_C})$.

9.1 - Solved Problem 5: Two computer chips are selected from a large shipment. Let $\mathbf{D_1}$ be the event that the first chip drawn is defective and $\mathbf{D_2}$ be the event that the second chip drawn is defective. Write $\mathbf{D_2}$ as an expression of $\mathbf{D_1}$ and $\mathbf{D_2}$.

Solution:

1. The first and second computer chips drawn are both defective: $\mathbf{D_1 \cap D_2}$.

or

2. The first computer chip drawn is not defective **but** the second computer chip drawn is defective: $\mathbf{D_1' \cap D_2}$.

3. Connecting these two event gives $\mathbf{D_2} = (\mathbf{D_1 \cap D_2}) \cup (\mathbf{D_1' \cap D_2})$.

Unsolved Problems with Answers

9.1 - Problems 1: Assume four computer chips are drawn from a box. If $\mathbf{D_j}$ is the event that the j th chip is defective, write out the event **E** that

(a). all of the chips are defective.

(b). only one chip is defective.

(c). none of the chips are defective.

(d). at least one chip is defective.

(e). Simplify $(\mathbf{D_1 \cap D_2 \cap D_3 \cap D_4}) \cap [\mathbf{D_1' \cup D_2' \cup D_3' \cup D_4'}]$.

(f). Simplify $(\mathbf{D_1 \cup D_2 \cup D_3 \cup D_4}) \cup [\mathbf{D_1' \cap D_2' \cap D_3' \cap D_4'}]$.

Answers:

➤ **(a).** $\mathbf{D_1 \cap D_2 \cap D_3 \cap D_4}$

➤ **(b)**. $(D_1 \cap D_2' \cap D_3' \cap D_4') \cup (D_1' \cap D_2 \cap D_3' \cap D_4') \cup (D_1' \cap D_2' \cap D_3 \cap D_4') \cup (D_1' \cap D_2' \cap D_3' \cap D_4)$

➤ **(c)**. $D_1' \cap D_2' \cap D_3' \cap D_4'$

➤ **(d)**. $D_1 \cup D_2 \cup D_3 \cup D_4$

➤ **(e)**. ϕ

➤ **(f)**. **S**

⇑ *Refer back to* **9.1 - Example 1 & 9.1 - Solved Problem 1.**

9.1 - Problem 2: A manufacturer of television sets purchases its television picture tubes from three different manufacturers, each located in different areas of the United States. Assume three picture tubes are selected at random, one at a time. Let E_i represent the event that the i th tube selected came from the East coast, W_i represent the event that the i th tube came from the West coast, and M_i represent the event that the i th tube came from the Midwest. Write out the expression for the event **F**:

(a). Two picture tubes selected came from the East coast and one from the West coast.

(b). Exactly two picture tubes selected came from the Midwest.

(c). No picture tubes selected came from the West coast.

(d). The first two picture tubes selected came from the west coast and the last selected came from the East coast.

(e). Each picture tube selected came from a different area.

Answers:

➤ **(a)**. $(E_1 \cap E_2 \cap W_3) \cup (E_1 \cap W_2 \cap E_3) \cup (W_1 \cap E_2 \cap E_3)$

➤ **(b)**. $(M_1 \cap M_2 \cap M_3') \cup (M_1 \cap M_2' \cap M_3) \cup (M_1' \cap M_2 \cap M_3)$

➤ **(c)**. $W_1' \cap W_2' \cap W_3'$

➤ **(d)**. $W_1 \cap W_2 \cap E_3$

➤ **(e)**. $(W_1 \cap E_2 \cap M_3) \cup (W_1 \cap M_2 \cap E_3) \cup (E_1 \cap W_2 \cap M_3) \cup (E_1 \cap M_2 \cap W_3) \cup (M_1 \cap W_2 \cap E_3) \cup (M_1 \cap E_2 \cap W_3)$

⇑ *Refer back to* **9.1 - Example 2 & 9.1 - Solved Problem 2**.

9.1 - Problem 3: The minimum speed on the San Diego freeway is 45 mph. Assume a car is observed traveling on this freeway at the minimum speed or faster, **S** = {45, 46, 47, 48,...}.

let **A** be the event it is traveling less than 75 mph.

Let **B** be the event it is traveling at least 47 mph .

Let **C** be the event it is traveling between 74 and 80 mph (inclusive). sing these events, find the following events **E**:

(a). the car is traveling at least 75 mph.

(b). the car is traveling 45 or 46 mph.

(c). it is traveling between 47 and 74 mph.

(d). it is traveling at exactly 74 mph.

Answers:

➤ **(a)**. **A′**

➤ **(b)**. **B′**

➤ **(c)**. **A∩B**

➤ **(d)**. **A∩C**

⇑ *Refer back to* **9.1 - Example 3 & 9.1 - Solved Problem 3.**

9.1 - Problem 4: A television manufacturer receives its television tubes from three independent suppliers. From a shipment of tubes, a tube was tested for flaws. Assume **T** represents the event that a tube is tested and found flawed. Let **A**, **B**, and **C** represent the events that the tube came from one of these suppliers respectively. Write **T** as an expression of **T**, **A**, **B**, **C**.

Answer:

T = (**T**∩**A**)∪(**T**∩**B**)∪(**T**∩**C**)

⇑ *Refer back to* **9.1 - Example 4 & 9.1 - Solved Problem 4.**

9.1 - Problem 1.5: A die is tossed twice. Let S_1 represent the event that a six occurs on the first toss and S_2 represent the event that a six occurs on the second toss. Write S_2 in terms of S_1 and S_2.

Answer:

$S_2 = (S_1 \cap S_2) \cup (S_1' \cap S_2)$

⇑ *Refer back to* **9.1 - Example 5 & 9.1 - Solved Problem 5**.

Supplementary Problems

1. Restate the DeMorgan laws for three sets **A**, **B**, **C**.

2. Four cards are selected from an ordinary deck. If $\mathbf{D}_i$ is the event that the i th card drawn is a diamond, simplify the event $\mathbf{E} = [(\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3 \cup \mathbf{D}_4)] \cap [\mathbf{D}_1 \cup (\mathbf{D}_2' \cap \mathbf{D}_3' \cap \mathbf{D}_4')]$.

3. A die is tossed until two 6s' occur or four times, whichever occurs first. If $\mathbf{A}_j$ (j = 1, 2, 3, 4) are the events that a six occurs on the j th toss, express the event **E** that

a. three tosses occurred.

b. there was at most three tosses.

c. there was four tosses.

4. Write the expression $\{(\mathbf{A}' \cap \mathbf{B}') \cup [\mathbf{C} \cup (\mathbf{D} \cap \mathbf{E})]' \cup [\mathbf{F}' \cap (\mathbf{G}' \cup \mathbf{H}')]\}'$ without complements.

For questions 5 & 6, apply the distributive laws:

5. $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C} \cup \mathbf{D} \cup \mathbf{E})$

6. $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C} \cap \mathbf{D} \cap \mathbf{E})$

For questions 7 & 8, prove the identity:

7. $(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{C} \cup \mathbf{D}) = (\mathbf{A} \cap \mathbf{C}) \cup (\mathbf{A} \cap \mathbf{D}) \cup (\mathbf{B} \cap \mathbf{C}) \cup (\mathbf{B} \cap \mathbf{D})$

8. $(\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{C} \cap \mathbf{D}) = (\mathbf{A} \cup \mathbf{C}) \cap (\mathbf{A} \cup \mathbf{D}) \cap (\mathbf{B} \cup \mathbf{C}) \cap (\mathbf{B} \cup \mathbf{D})$

9. Assume $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{C} \subseteq \mathbf{D} \subseteq \mathbf{E}$ then$(\mathbf{B} \cap \mathbf{A}') \cup (\mathbf{C} \cap \mathbf{B}') \cup (\mathbf{D} \cap \mathbf{C}') \cup (\mathbf{E} \cap \mathbf{D}')$ can be simplified to the intersection of two sets. Find this intersection.

10. An urn contains 6 red marbles and 4 blue marbles. Assume three marbles are drawn from the urn at random. Let $\mathbf{R}_i$ be the event that the ith (i = 1, 2, 3) marble drawn is red. For problems a-d, express the events in terms of $\mathbf{R}_i$, unions, intersection, and complements.

a. Exactly two of the marbles drawn are blue.

b. At most two of the marbles drawn are red.

c. Exactly two of the marbles are the same color.

d. A marble drawn on the second drawing is red.

11.Assume **C** is the event that a person will have a hamburger and **D** is the event that a person will have a frankfurter at a drive-in restaurant. Using **C**, **D**, unions, intersections and complements, express the following events:

a. a person will order either a hamburger or a frankfurter at this restaurant.

b. a person will order neither a hamburger nor a frankfurter at this restaurant.

c. a person will not order a hamburger at this restaurant.

d. a person will order a hamburger but not a frankfurter.

e. a person will order a frankfurter but not a hamburger.

f. a person will order one or the other but not both.

12. In a certain town, consider the events that a driver will receive one, two, three, four, or five or more tickets within one year. Assume **T1**, **T2**, **T3**, **T4**, **T5** represent each of these events respectively. Using these events as well as unions, intersections, and complements, express the event the driver, within one year will:

a. receive one or two traffic citations.

b. receive at most one traffic citation.

c. receive at least three traffic citations.

d. receive no traffic citations.

e. not get three traffic citations.

13. Ms. Jones loves to talk on the phone. Let Ck be the event that on a given day, she makes at least k phone calls (k=1, 2, 3, 4, 5,...). Express in your own words the meaning of the following events:

a. $\mathbf{C1}$

b. $\mathbf{C5}'$

c. $\mathbf{C7} \cap \mathbf{C8}'$

d. $\mathbf{C10} \cup \mathbf{C1}'$

e. $\mathbf{C10}' \cap \mathbf{C9}$

f. $\mathbf{C10}' \cup \mathbf{C12}$

g. $\mathbf{C1}'$

h. $(\mathbf{C8}' \cup \mathbf{C15})'$

14. Three cards are drawn from a 52 card deck. Let $\mathbf{K_i}$ be the event that the ith card drawn is a king, let $\mathbf{Q_i}$ be the event that the ith card drawn is a queen; and let $\mathbf{J_i}$ be the event that the ith card drawn is a jack (i=1, 2, 3). Using unions, intersections and complements, express the following events:

a. all cards drawn are kings.

b. no cards drawn are kings.

c. at least two cards drawn are Queens.

d. exactly one Jack is drawn.

e. two cards drawn are Queens and one card is a Jack.

f. exactly two cards drawn are Queens.

g. exactly two cards drawn are Queens and the other is a Jack or a King.

h. the first two cards drawn are kings.

i. no face cards are drawn.

If **F** is the event that a student will get financial aid, **J** is the event that he will find a part-time job, and **G** is the event that he will graduate, express the following events:

15. A student who gets financial aid will also graduate or get a part-time job.

16. A student who gets financial aid will not graduate and will not get a part-time job.

17. A student will not graduate or will get a part-time job but will not get financial aid.

A machine is producing ball bearings. Each hour, three ball bearings are selected at random, one at a time. Let $\mathbf{D_i}$ (i = 1, 2, 3) represent the event that the i th ball bearing is defective. Write out the expression for the event **E**:

18. two ball bearings are defective.

19. at least one ball bearings is defective.

20. no ball bearings are defective.

21. the first two drawn are defective and the last drawn is not defective.

Assume three cards are drawn from an ordinary deck of cards. If $\mathbf{K_i}$ (i = 1, 2, 3) are the events that the ith card drawn is a king and $\mathbf{Q_i}$ (i = 1, 2, 3) are the events that the i th card drawn is a queen. Write out the event **E** that

22. no king or queen is drawn.

23. one card is a king and two are queens.

24. only one king and queen is selected.

25. only one queen and no kings are selected.

26. Two urns, containing several red and white marbles are sitting on a table. A marble is selected from the first urn and placed in the second urn. Then a marble is selected from the second urn. Let $\mathbf{R_j}$ (j = 1, 2) represent the event that the marble selected from urn j is red. Write $\mathbf{R_2}$ as an expression of $\mathbf{R_1}$ **and** $\mathbf{R_2}$.

A recent survey of men was taken to find out their participation in the following sports: football, baseball, and ice hockey. A member of this group is selected. Let **B** represent the event that he plays baseball, **F** represent the event that he plays football, and **I** represent the event that he plays ice hockey. Using these events along with unions, intersections and complements, find the following events 27 - 31 :

27. he only plays football and ice hockey.

28. he only plays baseball.

29. he only plays one of these sports.

30. he does not play any of these sports.

31. Express the event $(\mathbf{F}\cap\mathbf{I}')\cup(\mathbf{F}'\cap\mathbf{I})$.

32. Show $(\mathbf{A}\cup\mathbf{B})\cap(\mathbf{C}\cup\mathbf{D}) = (\mathbf{A}\cap\mathbf{C})\cup(\mathbf{A}\cap\mathbf{D})\cup(\mathbf{B}\cap\mathbf{C})\cup(\mathbf{B}\cap\mathbf{D})$.

33. Using the laws on sets, prove the following:

a. **Uniqueness of the complement:**

If $\mathbf{B}\cap\mathbf{A} = \phi$ and $\mathbf{B}\cup\mathbf{A} = \mathbf{S}$ then $\mathbf{B} = \mathbf{A}'$.

b. $\mathbf{A}'' = \mathbf{A}$

c. $\mathbf{A}\cap\mathbf{A} = \mathbf{A}$

d. $\mathbf{A}\cup\mathbf{A} = \mathbf{A}$

e. $\mathbf{S}\cup\mathbf{A} = \mathbf{S}$

f. $\mathbf{A}\cap\phi = \phi$

g. $\phi' = \mathbf{S}$

h. $\mathbf{S}' = \phi$

i. **DeMorgan Laws:**

$(\mathbf{A}\cup\mathbf{B})' = \mathbf{A}'\cap\mathbf{B}'$

$(\mathbf{A}\cap\mathbf{B})' = \mathbf{A}'\cup\mathbf{B}'$

j. **Anti-symmetric Law:** If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$ then $\mathbf{A} = \mathbf{B}$.

k. If $\mathbf{A}\subseteq\mathbf{B}$ then $\mathbf{B} = \mathbf{B}\cup\mathbf{A}$.

l. If $\mathbf{A}\subseteq\mathbf{B}$ then $\mathbf{A} = \mathbf{B}\cap\mathbf{A}$.

m. **Transitive Law:** If $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{C}$ then $\mathbf{A} \subseteq \mathbf{C}$.

n. **Complement Law:** If $\mathbf{A}\subseteq \mathbf{B}$ then $\mathbf{B}'\subseteq \mathbf{A}'$.

o. Assume **A** is an arbitrary set. Show that the null set $\phi \subseteq \mathbf{A}$.

34. A disjoint partition of $\mathbf{A}\cup\mathbf{B}$ is $\mathbf{A}\cup\mathbf{B} = (\mathbf{A}\cap\mathbf{B}')\cup(\mathbf{A}'\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{B})$.

a. Find a disjoint partition of $\mathbf{A}\cup\mathbf{B}\cup\mathbf{C}$, (Hint: Use a Venn diagram).

b. Assume

A = {1,2,3,4, a,g,m,d,j,

1,2,3,4,b,k,n,e,h,

C = {1,2,3,4,i,c,l,f,o,

Find the 7 subsets that make up the partition of $\mathbf{A}\cup\mathbf{B}\cup\mathbf{C}$.

35. Using the laws of boolean algebra, show $(\mathbf{B}\cap\mathbf{C})\cup(\mathbf{B}\cap\mathbf{C}')\cup(\mathbf{B}'\cap\mathbf{C})\cup(\mathbf{B}'\cap\mathbf{C}') = \mathbf{S}$.

36. Using the laws of boolean algebra, show $(\mathbf{A}\cap\mathbf{B}\cap\mathbf{C})\cup(\mathbf{A}\cap\mathbf{B}\cap\mathbf{C}')\cup(\mathbf{A}\cap\mathbf{B}'\cap\mathbf{C})\cup(\mathbf{A}\cap(\mathbf{B}'\cap\mathbf{C}') = \mathbf{A}$.

37. Show if $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{A} \subseteq \mathbf{B}\cup\mathbf{C}$.

38. If all proper subsets of a set **A** are subsets of a set **B** show **A** is a subset of **B**.

39. Assume 3 cards are drawn from an ordinary deck of cards. Let $\mathbf{K}_i$ (i = 1,2,3) the event that the i th card drawn is a king. Let

$\mathbf{E} = (\mathbf{K}_1\cap\mathbf{K}_2\cap\mathbf{K}_3)\cup(\mathbf{K}_1'\cap\mathbf{K}_2\cap\mathbf{K}_3)\cup(\mathbf{K}_1\cap\mathbf{K}_2'\cap\mathbf{K}_3)\cup(\mathbf{K}_1\cap\mathbf{K}_2\cap\mathbf{K}_3')\cup(\mathbf{K}_1'\cap\mathbf{K}_2'\cap\mathbf{K}_3)\cup(\mathbf{K}_1'\cap\mathbf{K}_2\cap\mathbf{K}_3')\cup$
$(\mathbf{K}_1\cap\mathbf{K}_2'\cap\mathbf{K}_3')\cup(\mathbf{K}_1'\cap\mathbf{K}_2'\cap\mathbf{K}_3')$
Show $\mathbf{E} = \mathbf{S}$.

40. Assume in a family of sets the following is true: if $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{B} \subseteq \mathbf{A}$. Show this family of sets only contains the empty set ϕ.