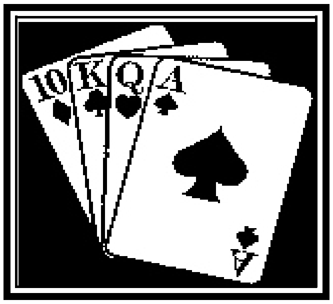


# Probability Theory
# Lesson 12
# Conditional Probability

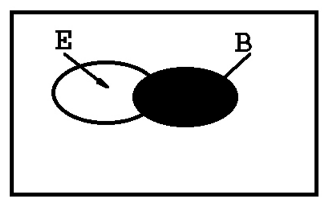


Frequently the probability of an event can be changed by having an a priori or previous knowledge of other events. For example, the probability of a 4 appearing in the toss of a single die is 1/6. However, if it is known or assumed that an even number resulted from the experiment, then the probability of a 4 appearing changes to 1/3. since we know that an odd number could not occur. Another example is the probability of rain on Tuesday would be influenced if it is known that it rained the previous Monday.

## 12.1 - What is Conditional Probability?

The conditional probability of an event **E** is the probability of the event **E** given the a priori knowledge of the occurrence of one or more events. Assume we have a sample space where each single outcome has equal chance of occurring. In this sample space, we assume that the event **B** has occurred. The shaded area represents that portion of the event **E** that can happen. Knowing that **B** occurred precludes the non-shaded area from happening. Therefore, the conditional probability of the event **E** is

$$P(\mathbf{E}|\mathbf{B}) = \frac{\#(\mathbf{E}\cap\mathbf{B})}{\#\mathbf{B}}.$$




However, by dividing the numerator and the denominator by #**S**, we get the fraction

$$\frac{\#(\mathbf{E}\cap\mathbf{B})}{\#\mathbf{B}} = \frac{\frac{\#(\mathbf{E}\cap\mathbf{B})}{\#\mathbf{S}}}{\frac{\#\mathbf{B}}{\#\mathbf{S}}} = \frac{P(\mathbf{E}\cap\mathbf{B})}{P(\mathbf{B})}.$$

Using the above explanation We define, for any sample space, conditional probability of an event **E** given that the event **B** occurred as

$$P(\mathbf{E}|\mathbf{B}) = \frac{P(\mathbf{E}\cap\mathbf{B})}{P(\mathbf{B})}.$$

Rule 1: $P(\mathbf{E}'|\mathbf{B}) = 1 - P(\mathbf{E}|\mathbf{B})$.

Rule 2: $P(\mathbf{C}\cup\mathbf{D}|\mathbf{B}) = P(\mathbf{C}|\mathbf{B}) + P(\mathbf{D}|\mathbf{B}) - P(\mathbf{C}\cap\mathbf{D}|\mathbf{B})$.

Throughout these lessons, sample spaces generated by drawing cards from an ordinary deck of cards provides a rich assortment of problems that can be explored for acquiring a deeper understand of the nature of the subject and developing methods for solving complex problems. The following presents the individual cards that make up a 52 deck of ordinary deck of cards:

**Suites(4):**

**Diamonds (13):** ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
**Hearts (13):** ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
**Clubs (13):** ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
**Spades (13):** ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

**Face cards (12):**

**Diamonds (3):** jack, queen, king
**Hearts (3):** jack, queen, king
**Clubs (3):** jack, queen, king
**Spades (3):** jack, queen, king

**12.1 - Example 1:** Let **M** be the event that it will rain on Monday and **T** the event that it will rain on Tuesday. Assume that the chance it will rain on Monday is 0.20 and that the chance it will rain on both Monday and Tuesday is 0.10. If it rains on Monday, find the probability that it will rain on Tuesday.

**Solution:**

**Step 1:** **M**: The event that it will rain on Monday.

**T**: The event that it will rain on Tuesday.

**M∩T:** The event that it will rain on both Monday and Tuesday.

**Step 2:** $P(\mathbf{M}) = 0.20$

**Step 3**: $P(\mathbf{M} \cap \mathbf{T}) = 0.10$

**Step 4**: $P(\mathbf{T}|\mathbf{M}) = \dfrac{P(\mathbf{M} \cap \mathbf{T})}{P(\mathbf{M})} = \dfrac{0.10}{0.20} = \dfrac{1}{2}$

**12.1- Example 2:** One card is randomly selected from an ordinary deck of cards.

(a). If a king is drawn, find the probability that the king is a diamond: $P(\mathbf{D}|\mathbf{K})$.

(b). If a king is not drawn, find the probability that the card is a diamond: $P(\mathbf{D}|\mathbf{K}')$.

**Solutions:**

➤**(a)**.
**Step 1:** There are 4 kings in an ordinary deck of cards.

**Step 2:** There is only 1 king that is diamond.

Therefore, $P(\mathbf{D}|\mathbf{K}) = 1/4$.

➤ **(b)**.
**Step 1:** There are 48 cards that are not kings in an ordinary deck of cards.

**Step 2:** An ordinary deck of cards has 13 diamonds of which 12 of them are not kings.

Therefore, $P(\mathbf{D}|\mathbf{K}') = 12/48 = 1/4$.

**12.1 - Example 3:** Two cards are drawn at random, without replacement from an ordinary deck of cards.

(a). If the first card drawn is a king, find the probability that the second card drawn is also a king: $P(\mathbf{K}_2|\mathbf{K}_1)$.

(b). If the first card drawn is a king, find the probability that the second card drawn is not a king: $P(\mathbf{K}_2'|\mathbf{K}_1)$.

(c). If the first card drawn is not a king, find the probability that the second card drawn is a king: $P(\mathbf{K}_2|\mathbf{K}_1')$.

(d). If the first card drawn is not a king, find the probability that the second card drawn is also not a king: $P(\mathbf{K}_2'|\mathbf{K}_1')$.

(e). If the first card drawn is a king, find the probability that the second card drawn is a queen: $P(\mathbf{Q}_2|\mathbf{K}_1)$.

**Solutions:**

There are 52 cards in an ordinary deck of cards of which 4 are kings cards.

➤ **(a)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is a king, there are only 3 kings remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are only 3 kings remaining in the deck, the probability that the second card drawn is a king is $P(\mathbf{K}_2|\mathbf{K}_1) = 3/51$.

➤ **(b)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is a king, there are 48 non-kings remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 48 non-kings remaining in the deck, the probability that the second card drawn is not a king is P($\mathbf{K}_2'|\mathbf{K}_1$) = 48/51.

➤ **(c)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is not a king, there are 4 kings remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 4 kings remaining in the deck, the probabiity that the second card drawn is a king is P($\mathbf{K}_2\,|\mathbf{K}_1'$) = 4/51.

➤ **(d)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is not a king, there are 47 non-kings remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 47 non-kings remaining in the deck, the probability that the second card drawn is not a king is P($\mathbf{K}_2'\,|\mathbf{K}_1'$) = 47/51.

➤ **(e)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is a king, there are 4 queens remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 4 queens remaining in the deck, the probabiity that the second card drawn is not queen is ($\mathbf{Q}_2\,|\mathbf{K}_1$) = 4/51.

## Solved Problems

**12.1 - Solved Problem 1:** Let **B** be the event that Mr. Jones is stopped for speeding and **E** the event that he is driving under the influence. Assume the chance that he will be stopped for speeding is 0.35 and that the chance he is stopped for speeding and at the same time driving under the influence is 0.25. If he is stopped for speeding, find the probability that he is also driving under the influence.

**Solution:**

**Step 1:** P(**B**) = 0.35.

**Step 2:** P(**E**∩**B**) = 0.25

**Step 3:** $P(\mathbf{E}|\mathbf{B}) = \dfrac{P(\mathbf{E}\cap\mathbf{B})}{P(\mathbf{B})} = \dfrac{0.25}{0.35} = \dfrac{5}{7}$

**12.1- Solved Problem 2:** One card is randomly selected from an ordinary deck of cards.

(a). If a face card is drawn, find the probability that the card is a queen: P(**Q**|**F**).

(b). If a face card is not drawn, find the probability that the card is a queen: P(**Q**|**F**′).

**Solutions:**

➤**(a)**.
**Step 1:** There are 12 face cards in an ordinary deck of cards.

**Step 2:** There are only 4 queens in the deck.

Therefore, P(**Q**|**F**) = 4/12 = 1/3.

➤**(b)**.
Since a non-face card is selected, and a queen is a face card, P(**Q**|**F**′) = 0/40 = 0.

**12.1 - Solved Problem 3:** Two cards are drawn at random, without replacement from an ordinary deck of cards.

(a). If the first card drawn is a face card, find the probability that the second card drawn is also a face card: P($\mathbf{F}_2$|$\mathbf{F}_1$).

(b). If the first card drawn is a face card, find the probability that the second card drawn is not a face card: P($\mathbf{F}_2$′|$\mathbf{F}_1$).

(c). If the first card drawn is not a face card, find the probability that the second card drawn is a face card: P($\mathbf{F}_2$ |$\mathbf{F}_1$′).

(d). If the first card drawn is not a face card, find the probability that the second card drawn is also not a face card: P($\mathbf{F}_2$′ |$\mathbf{F}_1$′).

(e). If the first card drawn is not a face card, find the probability that the second card drawn is a queen: P($\mathbf{Q}_2$ |$\mathbf{F}_1$′).

**Solutions:**

There are 52 cards in an ordinary deck of cards of which 12 are face cards.

➤**(a)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is a face card, there are only 11 face cards remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are only 11 face cards remaining in the deck, the probability that the second card drawn is a face card is P($\mathbf{F}_2$|$\mathbf{F}_1$) = 11/51.

➤**(b)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is a face card, there are 40 (52 - 12) non-face cards remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 40 non-face cards remaining in the deck, the probability that the second card drawn is not a face card is P($\mathbf{F}_2'|\mathbf{F}_1$) = 40/51.

➤**(c)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is not a face card, there are 12 face cards remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 12 face cards remaining in the deck, the probability that the second card drawn is a face card is P($\mathbf{F}_2\,|\mathbf{F}_1'$) = 12/51.

➤**(d)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is not a face card, there are 39 non-face cards remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 39 non-face cards remaining in the deck, the probability that the second card drawn is not a face card is P($\mathbf{F}_2'\,|\mathbf{F}_1'$) = 39/51.

➤**(e)**.
**Step 1:** After the first card is drawn, there are 51 cards remaining in the deck.

**Step 2:** Since the first card drawn is not a face card, there are 4 queens remaining in the deck.

**Step 3:** Since the remaining deck has only 51 cards and there are 4 queens remaining in the deck, the probabiity that the second card drawn is not queen is ($\mathbf{Q}_2\,|\mathbf{F}_1'$) = 4/51.

## Unsolved Problems with Answers

**12.1 - Problem 1:** Mr. Jones was recently murdered. Let **A** be the event that Mrs. Jones is arrested for the murder and **G** the event that she is guilty. Assume the chance that she will be arrested is 0.75 and the chance she is arrested and also guilty is 0.20. Find the probability that if she is arrested, then she is guilty.

**Answer:**

4/15

⇑ *Refer back to* **12.1 - Example 1 & 12.1 - Solved Problem 1**.

**12.1- Problem 2:** One card is randomly selected from an ordinary deck of cards.

(a). If a diamond is drawn, find the probability that the card is a face card: P(**F**|**D**).

(b). If a diamond is not drawn, find the probability that the card is a face card: P(**F**|**D**′).

**Answers:**

➤**(a)**. 3/13

➤**(b)**. 3/13

⥣ *Refer back to* **12.2 - Example 2 & 12.1 - Solved Problem 2**.

**12.1 - Problem 3:** Two cards are drawn at random, without replacement from an ordinary deck of cards.

(a). If the first card drawn is a diamond, find the probability that the second card drawn is also a diamond: $P(\mathbf{D}_2|\mathbf{D}_1)$.

(b). If the first card drawn is a diamond, find the probability that the second card drawn is not a diamond: $P(\mathbf{D}_2'|\mathbf{D}_1)$.

(c). If the first card drawn is not a diamond, find the probability that the second card drawn is a diamond card: $P(\mathbf{D}_2\,|\mathbf{D}_1')$.

(d). If the first card drawn is not a diamond, find the probability that the second card drawn is also not a diamond: $P(\mathbf{D}_2'\,|\mathbf{D}_1')$.

(e). If the first card drawn is a diamond, find the probability that the second card drawn is a heart :
$P(\mathbf{H}_2\,|\mathbf{D}_1)$.

**Answers:**

➤**(a)**. 12/51

➤**(b)**. 39/51

➤**(c)**. 13/51

➤**(d)**. 38/51

➤**(e)**. 13/51

⥣ *Refer back to* **12.1 - Example 3 & 12.1 - Solved Problem 3**.

# 12.2 - A Formula for P(E∩B)

From the formula $P(\mathbf{E}|\mathbf{B}) = \dfrac{P(\mathbf{E}\cap\mathbf{B})}{P(\mathbf{B})}$, we can derive

$$P(\mathbf{A}\cap\mathbf{B}) = P(\mathbf{A})P(\mathbf{B}|\mathbf{A})$$
or
$$P(\mathbf{A}\cap\mathbf{B}) = P(\mathbf{B})P(\mathbf{A}|\mathbf{B})$$

by multiplying both sides by the denominator.

This formula can be generalized as

$$P(\mathbf{E}_1 \cap \mathbf{E}_2 \cap \mathbf{E}_3 \cap \ldots \cap \mathbf{E}_n) = P(\mathbf{E}_1)P(\mathbf{E}_2|\mathbf{E}_1)P(\mathbf{E}_3|\mathbf{E}_1\cap\mathbf{E}_2)P(\mathbf{E}_4 \,|\mathbf{E}_1\cap\mathbf{E}_2\cap\mathbf{E}_3)\ldots P(\mathbf{E}_n \,|\mathbf{E}_1\cap\mathbf{E}_2\cap\mathbf{E}_3\ldots\cap\mathbf{E}_{n-1}).$$

**12.2 - Example 1:** Assume two cards are drawn from an ordinary deck of cards without replacement. Find the probability that both cards drawn are kings.

**Solution:**

**Step 1:** $\mathbf{K}_1$ : the event that a king is drawn on the first drawing.

**Step 2:** $\mathbf{K}_2$ : the event that a king was also drawn on the second drawing.

**Step 3:** $\mathbf{K}_1\cap\mathbf{K}_2$ : the event that both cards drawn are kings.

**Step 4:** $\#\mathbf{K}_1 = 4$

$P(\mathbf{K}_2 \,|\mathbf{K}_1) = \frac{3}{51}$

Thus, $P(\mathbf{K}_1\cap\mathbf{K}_2) = P(\mathbf{K}_1)P(\mathbf{K}_2|\mathbf{K}_1) = (\frac{4}{52})(\frac{3}{51}) = \frac{12}{2652} = \frac{1}{221}$.

**12.2 - Example 2:** Assume three cards are drawn from an ordinary deck of cards without replacement. Find the probability that

(a). all three cards drawn are kings.

(b). at least one king is drawn.

**Solutions:**

➤**(a)**.
$\mathbf{K}_1$: King is drawn on the first drawing.
$\mathbf{K}_2$: King is drawn on the second drawing.
$\mathbf{K}_3$: King is drawn on the third drawing.

$\mathbf{E} = \mathbf{K}_1\cap\mathbf{K}_2\cap\mathbf{K}_3$: all three cards drawn are kings.
Using the above formula, we have $P(\mathbf{E}) = P(\mathbf{K}_1)P(\mathbf{K}_2|\mathbf{K}_1)P(\mathbf{K}_3 \mid \mathbf{K}_1\cap\mathbf{K}_2) = (\frac{4}{52})(\frac{3}{51})(\frac{2}{50}) = \frac{24}{132600}$.

➤**(b)**.
Let $\mathbf{E}$ be the event that at least one king is drawn.

**Step 1:** $\mathbf{E} = \mathbf{K}_1\cup\mathbf{K}_2\cup\mathbf{K}_3$

Since we don't have a simple formula for the union of more than 3 sets (see Lesson 6, supplementary problem

12),we first compute

**Step 2:** $\mathbf{E}' = (\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3)' = \mathbf{K}_1' \cap \mathbf{K}_2' \cap \mathbf{K}_3'$, no kings are drawn.

**Step 3:** $P(\mathbf{E}') = P(\mathbf{K}_1')P(\mathbf{K}_2' | \mathbf{K}_1')P(\mathbf{K}_3' \mid \mathbf{K}_1' \cap \mathbf{K}_2') = (\frac{48}{52})(\frac{47}{51})(\frac{46}{50}) = \frac{103776}{132600}$

**Step 4:** $P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - \frac{103776}{132600} \approx 0.22$

**12.2 - Example 3:** An urn contains 10 white, 15 blue and 25 red marbles. Assume three marbles are selected from the urn without replacement. Find the probability that all three marbles selected are red.

**Solution:**

$\mathbf{R}_1$**:** The event that the marble drawn is red.

$\mathbf{R}_2$**:** The event that the second marble drawn is red.

$\mathbf{R}_3$**:** The event that the third marble drawn is red.

**E:** The event that all three marbles drawn are red is $\mathbf{E} = \mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3$.

**Step 1:** The probability that the first marble drawn is red is

$P(\mathbf{R}_1) = \frac{25}{50}$.

**Step 2:** Given that the first marble drawn is red, the probability that the second marble drawn is also red equals is

$P(\mathbf{R}_2 | \mathbf{R}_1) = \frac{24}{49}$.

**Step 3:** Given that the first and second marbles drawn are red, the probability that the third marble drawn is also red is

$P(\mathbf{R}_3 | \mathbf{R}_1 \cap \mathbf{R}_2) = \frac{23}{48}$.

**Step 4:** Therefore, the probability that the first, second and third marbles are all red is $P(\mathbf{E}) = P(\mathbf{R}_1 \cap \mathbf{R}_2 \cap \mathbf{R}_3) =$

$$(\frac{25}{50})(\frac{24}{49})(\frac{23}{48}) = \frac{13800}{117600} = \frac{23}{196}.$$

## Solved Problems

**12.2 - Solved Problem 1:** Assume two cards are drawn from an ordinary deck of cards without replacement. Find the probability that the first card drawn is a king and the second card drawn is an ace.

**Solution:**

Let $\mathbf{K}_1$ be the event that a king is drawn on the first drawing and $\mathbf{A}_2$ the event that an ace was drawn on the second drawing.

Thus, $P(\mathbf{K}_1 \cap \mathbf{A}_2) = P(\mathbf{K}_1)P(\mathbf{A}_2|\mathbf{K}_1) = (\frac{4}{52})(\frac{4}{51}) = \frac{16}{2652} = \frac{4}{663}$.

**12.2 Solved Problem 2:** Assume three cards are drawn from an ordinary deck of cards without replacement. Find the probability that

(a). all three cards are face cards.

(b). at least one face card is drawn.

**Solutions:**

➤**(a)**.
$\mathbf{F}_1$: Face card is drawn on the first drawing.
$\mathbf{F}_2$: Face card is drawn on the second drawing.
$\mathbf{F}_3$: Face card is drawn on the third drawing.

$\mathbf{E} = \mathbf{F}_1 \cap \mathbf{F}_2 \cap \mathbf{F}_3$: all three cards drawn are face cards.

Using the above formula, we have $P(\mathbf{E}) = P(\mathbf{F}_1)P(\mathbf{F}_2|\mathbf{F}_1)P(\mathbf{F}_3 \mid \mathbf{F}_1 \cap \mathbf{F}_2) = (\frac{12}{52})(\frac{11}{51})(\frac{10}{50}) = \frac{1320}{132600}$.

➤**(b)**.
Let **E** be the event that at least one face card is drawn.

**Step 1:** $\mathbf{E} = \mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3$

Since we don't have a simple formula for the union of more than 3 sets (see Lesson 6, supplementary problem 12),

**Step 2:** we first compute $\mathbf{E}' = (\mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3)' = \mathbf{F}_1' \cap \mathbf{F}_2' \cap \mathbf{F}_3'$, no face cards are drawn.

**Step 3:** $P(\mathbf{E}') = P(\mathbf{F}_1')P(\mathbf{F}_2'|\mathbf{F}_1')P(\mathbf{F}_3' \mid \mathbf{F}_1' \cap \mathbf{F}_2') = (\frac{40}{52})(\frac{39}{51})(\frac{38}{50}) = \frac{59280}{132600}$

**Step 4:** $P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - (\frac{40}{52})(\frac{39}{51})(\frac{38}{50}) = \frac{73320}{132600} \approx 0.55$

**12.2 - Solved Problem 3:** An urn contains 10 white, 15 blue and 25 red marbles. Assume three marbles are selected from the urn without replacement. Find the probability that the first marble selected is red, the second marble selected is blue and third marble selected is white.

**Solution:**

$\mathbf{R}_1$: The event that the marble drawn is red.

$\mathbf{B}_2$: The event that the second marble drawn is blue.

$\mathbf{W}_3$: The event that the third marble drawn is white.

**E:** The event that the first marble selected is red, the second marble selected is blue and third marble selected is white: $\mathbf{E} = \mathbf{R}_1 \cap \mathbf{B}_2 \cap \mathbf{W}_3$.

**Step 1:** The probability that the first marble drawn is red is

$$P(\mathbf{E}) = \frac{25}{50}.$$

**Step 2:** Given that the first marble drawn is red, the probability that the second marble drawn is blue is

$$P(\mathbf{B}_2 | \mathbf{R}_1) = \frac{15}{49}.$$

**Step 3:** Given that the first marble drawn is red, the second marble drawn is blue, the probability that the third marble drawn is white is

$$P(\mathbf{W}_3 | \mathbf{R}_1 \cap \mathbf{B}_2) = \frac{10}{48}.$$

**Step 3:** Therefore the probability of **E** is

$$P(\mathbf{E}) = P(\mathbf{R}_1 \cap \mathbf{B}_2 \cap \mathbf{W}_3) = (\frac{25}{50})(\frac{15}{49})(\frac{10}{48}) = \frac{75}{2352}.$$

## Unsolved Problems with Answers

**12.2 - Problem 1:** Assume two cards are drawn from an ordinary deck of cards without replacement. Find the probability that the first card is a king and the second card is an ace or queen.

**Answer:**

8/663

⇑ *Refer back to* **12.2 - Example 1 & 12.2 - Solved Problem 1**.

**12.2 - Problem 2:** Assume three cards are drawn from an ordinary deck of cards without replacement. Find the probability that

(a). all three cards are clubs.

(b). at least one club is drawn.

**Answers:**

➤**(a)**. 1716/132600

➤**(b)**. 77766/132600

⇑ *Refer back to* **12.2 - Example 2 & 12.2 - Solved Problem 2**.

**12.2 - Problem 3:** An urn contains 10 white, 15 blue and 25 red marbles. Assume three marbles are selected from the urn without replacement. Find the probability that the first is red, the second is blue and third is red.

**Answer:**

15/196

⇑ *Refer back to* **12.2 - Example 3 & 12.2 - Solved Problem 3.**

**Answers:**

➤ **(a)**. $\frac{7}{102}$

➤ **(b)**. $\frac{35}{102}$

➤ **(c)**. $\frac{95}{102}$

⇑ *Refer back to* **12.2 - Example 4 & 12.2 - Solved Problem 4.**

# 12.3 - Writing the Event E in Terms of Other Events.

Under certain conditions, one event can be expressed in terms of other events.

The following formulas are important:

**Boolean algebra formulas:**

1. $\mathbf{E} = (\mathbf{E} \cap \mathbf{B}) \cup (\mathbf{E} \cap \mathbf{B}')$

2. If $\mathbf{E} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup \ldots \cup \mathbf{B}_n$ then $\mathbf{E}' = \mathbf{B}_1' \cap \mathbf{B}_2' \cap \ldots \cap \mathbf{B}_n'$

**Probability formulas:**

1a. $P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{B})P(\mathbf{A}|\mathbf{B})$

1b. $P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A})P(\mathbf{B}|\mathbf{A})$

2a. $P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \cap \mathbf{B})$

2b. If $\mathbf{A} \cap \mathbf{B} = \phi$ then $P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B})$

3a. $P(\mathbf{A}_1 \cup \mathbf{A}_2) = 1 - P(\mathbf{A}_1' \cap \mathbf{A}_2')$

3b. $P(\mathbf{A}_1 \cup \mathbf{A}_2 \cup ... \cup \mathbf{A}_n) = 1 - P(\mathbf{A}_1' \cap \mathbf{A}_2' \cap ... \cap \mathbf{A}_n')$

4. If $\mathbf{B}_i \cap \mathbf{B}_j = \phi$ $(1 \leq i,j \leq n)$ and $\mathbf{E} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup ... \cup \mathbf{B}_n$ then

$P(\mathbf{E}) = P(\mathbf{B}_1) + P(\mathbf{B}_2) + ... + P(\mathbf{B}_n)$

**Definition**: If $\mathbf{B}_i \cap \mathbf{B}_j = \phi$ $(1 \leq i,j \leq n)$, the sets $\mathbf{B}_i$ are said to be disjoint.

**12.3 - Example 1:** Two urns are sitting on a table. Urn 1 has 3 blue marbles and 7 white marbles. Urn 2 has 5 blue marbles and 10 white marbles. A coin is tossed once. If heads occurs a marble is selected at random from urn 1 and if tails occurs a marble is selected at random from urn 2. Find the probability that the marble selected is white.

**Solution:**

$\mathbf{U}_1$: the event that urn 1 is selected.

$\mathbf{U}_2$: the event that urn 2 is selected.

**W**: the event that a white marble is selected.

$\mathbf{W} \cap \mathbf{U}_1$**:** the event that a white is selected **AND** urn 1 is selected.

$\mathbf{W} \cap \mathbf{U}_2$**:** the event that a white is selected **AND** urn 2 is selected.

$\mathbf{W} = (\mathbf{W} \cap \mathbf{U}_1) \cup (\mathbf{W} \cap \mathbf{U}_2)$

To find $P(\mathbf{W})$, we do the following steps:

**Step1:** $P(\mathbf{W}) = P(\mathbf{W} \cap \mathbf{U}_1) + P(\mathbf{W} \cap \mathbf{U}_2)$

**Step2:** $P(\mathbf{W}\cap\mathbf{U_1}) = P(\mathbf{U}_1)P(\mathbf{W}|\mathbf{U}_1) = (\frac{1}{2})(\frac{7}{10}) = \frac{7}{20}$

**Step3:** $P(\mathbf{W}\cap\mathbf{U_2}) = P(\mathbf{U}_2)P(\mathbf{W}|\mathbf{U}_2) = (\frac{1}{2})(\frac{10}{15}) = \frac{1}{3}$

**Step4:** $P(\mathbf{W}) = P(\mathbf{W}\cap\mathbf{U}_1) + P(\mathbf{W}\cap\mathbf{U}_2) = \frac{7}{20} + \frac{1}{3} = \frac{41}{60}$

**12.3 - Example 2:** Two cards are drawn at random without replacement from an ordinary deck of cards.

(a). Find the probability that the second card drawn is a king.

(b). Find the probability that the first card is a king or the second card is a king.

(c). Find the probability that a king and queen are drawn.

(d). Find the probability that at least one king or one queen is drawn.

**Solutions:**

➤ **(a)**.
The chance of drawing a king on the second drawing depends on whether a king had been drawn on the first drawing.

$\mathbf{K}_1$: the event that a king is drawn on the first draw.

$\mathbf{K}_2$: the event that a king is drawn on the second drawing.

$\mathbf{K}_1\cap\mathbf{K}_2$ : the event that a king is drawn on the first **AND** second drawing.

$\mathbf{K}_1'\cap\mathbf{K}_2$: the event that a king is **NOT** drawn on the first **AND** a king is drawn on the second drawing.

$\mathbf{K}_2 = (\mathbf{K}_1\cap\mathbf{K}_2)\cup(\mathbf{K}_1'\cap\mathbf{K}_2)$: a king is drawn on first and on the second or a king is not drawn on the first but a king is drawn on the second.

$P(\mathbf{K}_2) = P(\mathbf{K}_1\cap\mathbf{K}_2) + P(\mathbf{K}_1'\cap\mathbf{K}_2) = (\frac{4}{52})(\frac{3}{51}) + (\frac{48}{52})(\frac{4}{51}) = \frac{204}{2652} = \frac{4}{52}$

➤ **(b)**.
$\mathbf{K}_1$: the event that a king is drawn on the first draw.

$\mathbf{K}_2$: the event that a king is drawn on the second draw.

**E:** the event that a king is drawn on the first or second drawing.

$\mathbf{E} = \mathbf{K}_1\cup\mathbf{K}_2$

$P(\mathbf{E}) = P(\mathbf{K}_1 \cup \mathbf{K}_2) = P(\mathbf{K}_1) + P(\mathbf{K}_2) - P(\mathbf{K}_1 \cap \mathbf{K}_2) = \frac{4}{52} + \frac{4}{52} - P(\mathbf{K}_1)P(\mathbf{K}_2 | \mathbf{K}_1) =$

$$\frac{4}{52} + \frac{4}{52} - (\frac{4}{52})(\frac{3}{51}) = \frac{408}{(52)(51)} - \frac{12}{(52)(51)} = \frac{33}{221}$$

*Alternative solution:*

$\mathbf{E}' = (\mathbf{K}_1 \cup \mathbf{K}_2)' = \mathbf{K}_1' \cap \mathbf{K}_2'$

$P(\mathbf{E}') = P(\mathbf{K}_1' \cap \mathbf{K}_2') = P(\mathbf{K}_1')P(\mathbf{K}_2' | \mathbf{K}_1') = (48/52)(47/51)$

$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - (48/52)(47/51) = 1 - 2256/2652 = 396/2652 = 33/221$

➤ **(c)**.
**E**: the event that a king and queen are drawn.

Since order is not required we must consider all possibilities: $\mathbf{E} = (\mathbf{K}_1 \cap \mathbf{Q}_2) \cup (\mathbf{Q}_1 \cap \mathbf{K}_2)$

$P(\mathbf{E}) = P(\mathbf{K}_1 \cap \mathbf{Q}_2) + P(\mathbf{Q}_1 \cap \mathbf{K}_2) = P(\mathbf{K}_1)P(\mathbf{Q}_2 | \mathbf{K}_1) + P(\mathbf{Q}_1)P(\mathbf{K}_2 | \mathbf{Q}_1) = (\frac{4}{52})(\frac{4}{51}) + (\frac{4}{52})(\frac{4}{51}) + \frac{32}{2652}$

➤ **(d)**.
**E**: at least 1 king or 1 queen is drawn.

$\mathbf{E} = (\mathbf{K}_1 \cup \mathbf{Q}_2) \cup (\mathbf{Q}_1 \cup \mathbf{K}_2)$

**E**′: no king and no queen are drawn.

$\mathbf{E}' = (\mathbf{K}_1' \cap \mathbf{Q}_2') \cap (\mathbf{Q}_1' \cap \mathbf{K}_2') = (\mathbf{K}_1' \cap \mathbf{Q}_1') \cap (\mathbf{K}_2' \cap \mathbf{Q}_2')$

$P(\mathbf{E}') = P[(\mathbf{K}_1' \cap \mathbf{Q}_1') \cap (\mathbf{K}_2' \cap \mathbf{Q}_2')] = P(\mathbf{K}_1' \cap \mathbf{Q}_1')P[(\mathbf{K}_2' \cap \mathbf{Q}_2') | (\mathbf{K}_1' \cap \mathbf{Q}_1')] = (\frac{44}{52})(\frac{43}{51}) = \frac{1892}{2652} = \frac{473}{663}$

$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - \frac{473}{663} = \frac{190}{663}$

**12.3 - Example 3:** Two cards are drawn at random without replacement from an ordinary deck of cards.

(a). Find the probability that the first card is a diamond and the second card is a king.

(b). Find the probability that the first card is a diamond or the second card is a king.

**Solutions:**

➤ **(a)**.
$\mathbf{D}_1$: the event that a diamond is drawn on the first draw.

$\mathbf{K}_1$**:** the event that a king is drawn on the first draw.

$\mathbf{K}_2$: the event that a king is drawn on the second draw.

$\mathbf{E} = \mathbf{D}_1 \cap \mathbf{K}_2$: the event that a diamond is drawn on the first drawing and a king on the second drawing.

$\mathbf{D}_1 = (\mathbf{D}_1 \cap \mathbf{K}_1) \cup (\mathbf{D}_1 \cap \mathbf{K}_1')$

$\mathbf{E} = \mathbf{D}_1 \cap \mathbf{K}_2 = [(\mathbf{D}_1 \cap \mathbf{K}_1) \cup (\mathbf{D}_1 \cap \mathbf{K}_1')] \cap \mathbf{K}_2 = [(\mathbf{D}_1 \cap \mathbf{K}_1) \cap \mathbf{K}_2] \cup [(\mathbf{D}_1 \cap \mathbf{K}_1') \cap \mathbf{K}_2]$

$P(\mathbf{E}) = P(\mathbf{D}_1 \cap \mathbf{K}_2) = P([(\mathbf{D}_1 \cap \mathbf{K}_1) \cap \mathbf{K}_2] \cup [(\mathbf{D}_1 \cap \mathbf{K}_1') \cap \mathbf{K}_2]) = P[(\mathbf{D}_1 \cap \mathbf{K}_1) \cap \mathbf{K}_2] + P[(\mathbf{D}_1 \cap \mathbf{K}_1') \cap \mathbf{K}_2] =$

$P(\mathbf{D}_1 \cap \mathbf{K}_1) P[\mathbf{K}_2 | (\mathbf{D}_1 \cap \mathbf{K}_1)] + P(\mathbf{D}_1 \cap \mathbf{K}_1') P[\mathbf{K}_2 | (\mathbf{D}_1 \cap \mathbf{K}_1')] = (\frac{1}{52})(\frac{3}{51}) + (\frac{12}{52})(\frac{4}{51}) = \frac{1}{52}$

➤ **(b)**.
$\mathbf{D}_1$: the event that a diamond is drawn on the first draw.

$\mathbf{K}_2$: the event that a king is drawn on the second draw.

**E:** the event that a diamond is drawn on the first or a king is selected on second drawing.

$\mathbf{E} = \mathbf{D}_1 \cup \mathbf{K}_2$

$P(\mathbf{K}_2) = 4/52$ (See **12.3 - Example 2**).

$P(\mathbf{E}) = P(\mathbf{D}_1 \cup \mathbf{K}_2) = P(\mathbf{D}_1) + P(\mathbf{K}_2) - P(\mathbf{D}_1 \cap \mathbf{K}_2) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52}$

**12.3-Example 4:** Three cards are drawn without replacement from an ordinary deck of cards. Let $\mathbf{K}_i$, $\mathbf{Q}_i$ be the events that a king or queen is drawn respectively on the ith drawing.

(a). Show $P(\mathbf{K}_1 \cap \mathbf{Q}_3) = P(\mathbf{K}_1 \cap \mathbf{Q}_2)$.

(b). Find the probability that at least 2 kings are drawn.

(c). Find the probability that at least 1 king or 1 queen is drawn.

(d). Find the probability that at least 1 king and 1 queen is drawn.

**Solutions:**

➤ **(a)**.
Step 1: $\mathbf{K}_1 \cap \mathbf{Q}_3 = \mathbf{K}_1 \cap (\mathbf{Q}_2 \cup \mathbf{Q}_2') \cap \mathbf{Q}_3 = (\mathbf{K}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3) \cup (\mathbf{K}_1 \cap \mathbf{Q}_2' \cap \mathbf{Q}_3)$, (distributive law)

Step 2: $P(\mathbf{K}_1 \cap \mathbf{Q}_3) = P(\mathbf{K}_1 \cap \mathbf{Q}_2 \cap \mathbf{Q}_3) + P(\mathbf{K}_1 \cap \mathbf{Q}_2' \cap \mathbf{Q}_3) = (4/52)(4/51)(3/50) + (4/52)(47/51)(4/50) = 800/132600$

Step 3: $P(\mathbf{K}_1 \cap \mathbf{Q}_2) = (4/52)(4/51) = (4/52)(4/51)(50/50) = 800/132600$

➤ **(b)**.
**E**: at least 2 kings equals exactly 2 kings or 3 kings:

Step 1: $\mathbf{E} = \{(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3') \cup (\mathbf{K}_1 \cap \mathbf{K}_2' \cap \mathbf{K}_3) \cup (\mathbf{K}_1' \cap \mathbf{K}_2 \cap \mathbf{K}_3)\} \cup (\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3)$, (These sets are disjoint.)

Step 2: $P(\mathbf{E}) = P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3') + P(\mathbf{K}_1 \cap \mathbf{K}_2' \cap \mathbf{K}_3) + P(\mathbf{K}_1' \cap \mathbf{K}_2 \cap \mathbf{K}_3) + P(\mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3)$

(4/52)(3/51)(48/50) + (4/52)(48/51)(3/50) + (48/52)(4/51)(3/50) + (4/52)(3/51)(2/50) = 1752/132600

➤ **(c)**.
Step 1: $\mathbf{E} = (\mathbf{K}_1 \cup \mathbf{Q}_1) \cup (\mathbf{K}_2 \cup \mathbf{Q}_2) \cup (\mathbf{K}_3 \cup \mathbf{Q}_3)$

Step 2: $\mathbf{E}' = [(\mathbf{K}_1 \cup \mathbf{Q}_1) \cup (\mathbf{K}_2 \cup \mathbf{Q}_2) \cup (\mathbf{K}_3 \cup \mathbf{Q}_3)]' = (\mathbf{K}_1' \cap \mathbf{Q}_1') \cap (\mathbf{K}_2' \cap \mathbf{Q}_2') \cap (\mathbf{K}_3' \cap \mathbf{Q}_3')$ (DeMorgan law)

Step 3: $\mathbf{E}' = P[(\mathbf{K}_1' \cap \mathbf{Q}_1') \cap (\mathbf{K}_2' \cap \mathbf{Q}_2') \cap (\mathbf{K}_3' \cap \mathbf{Q}_3')]$ = (44/52)(43/51)(42/50) = 79464/132600

Step 4: $P(\mathbf{E}) = 1 - P(\mathbf{E}')$ = 1 - 79464/132600 = 52136/132600

➤ **(d)**.
**E**: The event that at least 1 king and 1 queen is drawn

$\mathbf{E} = (\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) \cap (\mathbf{Q}_1 \cup \mathbf{Q}_2 \cup \mathbf{Q}_3)$

$\mathbf{E}' = \{(\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) \cap (\mathbf{Q}_1 \cup \mathbf{Q}_2 \cup \mathbf{Q}_3)\}' = (\mathbf{K}_1' \cap \mathbf{K}_2' \cap \mathbf{K}_3') \cup (\mathbf{Q}_1' \cap \mathbf{Q}_2' \cap \mathbf{Q}_3')$, (DeMorgan's law)

$P(\mathbf{E}') = P(\mathbf{K}_1' \cap \mathbf{K}_2' \cap \mathbf{K}_3') + P(\mathbf{Q}_1' \cap \mathbf{Q}_2' \cap \mathbf{Q}_3') - P\{(\mathbf{K}_1' \cap \mathbf{Q}_1') \cap (\mathbf{K}_2' \cap \mathbf{Q}_2') \cap (\mathbf{K}_3' \cap \mathbf{Q}_3')\} =$

(48/52)(47/51)(46/50) + (48/52)(47/51)(46/50) - (44/52)(43/51)(42/50) = 128088/132600

$P(\mathbf{E}) = 1 - P(\mathbf{E})$ = 1 - 128008/132600 = 4592/132600

**12.3 - Example 5:** Two urns are sitting on a table. Urn 1 has three blue marbles and 7 white marbles. Urn 2 has 4 blue marbles and 10 white marbles. A marble is selected from urn 1 and placed in urn 2. Next, a marble is selected from urn 2. Find the probability that the marble selected from urn 2 is white.

**Solution:**

The chance of selecting a white marble from urn 2 depends on whether a white marble was selected from urn 1 and placed in urn 2.

$\mathbf{W}_1$: the event that a white marble is drawn from urn 1 and placed in urn 2.

$\mathbf{W}_2$: the event that a white is drawn from urn 2.

$\mathbf{W}_1 \cap \mathbf{W}_2$ : the event that a white marble is drawn from urn 1 and urn 2.

$\mathbf{W}_1' \cap \mathbf{W}_2$ : the event that a white marble is **NOT** drawn from urn 1 and a white marble is drawn from urn 2.

$\mathbf{W_2} = (\mathbf{W_1}' \cap \mathbf{W_2}) \cup (\mathbf{W_1} \cap \mathbf{W_2})$, the event that a white is drawn from urn 1 and urn 2 **OR** the event that a white is **NOT** drawn from urn 1 **AND** a white is selected from urn 2

$$P(\mathbf{W_1} \cap \mathbf{W_2}) = P(\mathbf{W_1})P(\mathbf{W_2} | \mathbf{W_1}) = \left(\frac{7}{10}\right)\left(\frac{11}{15}\right)$$

$$P(\mathbf{W_1}' \cap \mathbf{W_2}) = P(\mathbf{W_1}')P(\mathbf{W_2} | \mathbf{W_1}') = \left(\frac{3}{10}\right)\left(\frac{10}{15}\right)$$

Therefore,

$$P(\mathbf{W_2}) = P(\mathbf{W_1} \cap \mathbf{W_2}) + P(\mathbf{W_1}' \cap \mathbf{W_2}) = \left(\frac{7}{10}\right)\left(\frac{11}{15}\right) + \left(\frac{3}{10}\right)\left(\frac{10}{15}\right) = \frac{77}{150} + \frac{30}{150} = \frac{107}{150}.$$

**12.3 - Example 6:** In a state where cars have to be tested for emissions of pollutants, 25% of all cars emit excessive amounts of pollutants. When tested, 99% of all cars that emit excessive amounts of pollutants will fail, but 17% of the cars that do not emit excessive amounts of pollutants will also fail. A car is selected at random. What is the probability that the car fails the test?

**Solution:**

Let **E** be the event that a car emits excessive amounts of pollutants.

Let **F** be the event that the car will fail the test.

$P(\mathbf{E})$ = the probability that a car will emit excessive pollutants = 0.25

$P(\mathbf{F} | \mathbf{E})$ = the probability that all cars that emit excessive amounts of pollutants will fail = 0.99

$P(\mathbf{F} | \mathbf{E}')$ = the probability that a car that does not emit excessive pollutants fails the test = 0.17

We now need to solve for $P(\mathbf{F})$.

**Step 1:** **F** is the event that a car fails the test **and** it emits excessive pollutants **or** a car fails the test and it does not emit excessive pollutants: $\mathbf{F} = (\mathbf{F} \cap \mathbf{E}) \cup (\mathbf{F} \cap \mathbf{E}')$.

**Step 2:** $P(\mathbf{F}) = P(\mathbf{F} \cap \mathbf{E}) + P(\mathbf{F} \cap \mathbf{E}') = P(\mathbf{E})P(\mathbf{F} | \mathbf{E}) + P(\mathbf{E}')P(\mathbf{F} | \mathbf{E}') = 0.25(0.99) + 0.75(0.17) = 0.375$

## Solved Problems

**12.3 - Solved Problem 1**: Two urns are sitting on a table. Urn 1 has five blue marbles, 10 red marbles and 15 white marbles. Urn 2 has 5 blue marbles and 10 white marbles. A coin is tossed twice. If heads occur twice, a marble is selected at random from urn 1; otherwise, a marble is selected at random from urn 2. Find the probability that the marble selected is white.

**Solution:**

Let $\mathbf{U}_1$ be the event that urn 1 is selected and let $\mathbf{U}_2$ be the event that urn 2 is selected. Let **W** be the event that a white marble is selected.

Here, $P(\mathbf{U}_1) = 1/4$.

To find the $P(\mathbf{W})$ we do the following steps:

**Step 1:** $P(\mathbf{W}) = P(\mathbf{W}\cap\mathbf{U}_1) + P(\mathbf{W}\cap\mathbf{U}_2)$

**Step 2:** $P(\mathbf{W}\cap\mathbf{U}_1) = P(\mathbf{U}_1)P(\mathbf{W}|\mathbf{U}_1) = (\frac{1}{4})(\frac{15}{30}) = \frac{1}{8}$

**Step 3:** $P(\mathbf{W}\cap\mathbf{U}_2) = P(\mathbf{U}_2)P(\mathbf{W}|\mathbf{U}_2) = (\frac{3}{4})(\frac{10}{15}) = \frac{1}{2}$

**Step 4:** $P(\mathbf{W}) = P(\mathbf{W}\cap\mathbf{U}_1)+P(\mathbf{W}\cap\mathbf{U}_2) = \frac{1}{8}+\frac{1}{2} = \frac{5}{8}$

**12.3 - Solved Problem 2:** Two cards are drawn at random without replacement from an ordinary deck of cards.

(a). Find the probability that the second card drawn is a diamond.

(b). Find the probability that the first card is a diamond or the second card is a diamond.

(c). Find the probability that a diamond and a club are drawn.

(d). Find the probability that at least one diamond or one club is drawn.

**Solutions:**

➤ **(a)**.
The chance of drawing a diamond on the second drawing depends on whether a diamond had been drawn on the first drawing.

$\mathbf{D}_1$: the event that a diamond is drawn on the first draw.

$\mathbf{D}_2$: the event that a diamond is drawn on the second drawing.

$\mathbf{D}_1\cap\mathbf{D}_2$ : the event that a diamond is drawn on the first **AND** second drawing.

$\mathbf{D}_1'\cap\mathbf{K}_2$: the event that a diamond is **NOT** drawn on the first **AND** a diamond is drawn on the second drawing.

$\mathbf{D}_2 = (\mathbf{D}_1\cap\mathbf{D}_2)\cup(\mathbf{D}_1'\cap\mathbf{D}_2)$: a diamond is drawn on first and on the second or a diamond is not drawn on the first but a diamond is drawn on the second**.**

$P(\mathbf{D}_2) = P(\mathbf{D}_1\cap\mathbf{D}_2) + P(\mathbf{D}_1'\cap\mathbf{D}_2) = (\frac{13}{52})(\frac{12}{51}) + (\frac{39}{52})(\frac{13}{51}) = \frac{663}{2652} = \frac{13}{52}$

➤ **(b)**.
$\mathbf{D}_1$: the event that a diamond is drawn on the first draw.

$\mathbf{D}_2$: the event that a diamond is drawn on the second draw.

**E:** the event that a diamond is drawn on the first or second drawing.
$\mathbf{E} = \mathbf{D}_1 \cup \mathbf{D}_2$

$$P(\mathbf{E}) = P(\mathbf{D}_1 \cup \mathbf{D}_2) = P(\mathbf{D}_1) + P(\mathbf{D}_2) - P(\mathbf{D}_1 \cap \mathbf{D}_2) = \frac{13}{52} + \frac{13}{52} - (\frac{13}{52})(\frac{12}{51}) = \frac{1170}{2652}$$

*Alternative solution*:

$\mathbf{E}' = (\mathbf{D}_1 \cup \mathbf{D}_2)' = \mathbf{D}_1' \cap \mathbf{D}_2'$

$P(\mathbf{E}') = P(\mathbf{D}_1' \cap \mathbf{D}_2') = P(\mathbf{D}_1')P(\mathbf{D}_2' | \mathbf{D}_1') = (39/52)(38/51)$

$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - (39/52)(38/51) = 1 - 1482/2652 = 1170/2652$

➤ **(c)**.
**E**: the event that a diamond and club are drawn.

Since order is not required we must consider all possibilities: $\mathbf{E} = (\mathbf{D}_1 \cap \mathbf{C}_2) \cup (\mathbf{C}_1 \cap \mathbf{D}_2)$

$$P(\mathbf{E}) = P(\mathbf{D}_1 \cap \mathbf{C}_2) + P(\mathbf{C}_1 \cap \mathbf{D}_2) = P(\mathbf{D}_1)P(\mathbf{C}_2 | \mathbf{D}_1) + P(\mathbf{C}_1)P(\mathbf{D}_2 | \mathbf{C}_1) = (\frac{13}{52})(\frac{13}{51}) + (\frac{13}{52})(\frac{13}{51}) + \frac{338}{2652}$$

➤ **(d)**.
**E**: at least 1 diamond or club is drawn.

$\mathbf{E} = (\mathbf{D}_1 \cup \mathbf{C}_2) \cup (\mathbf{C}_1 \cup \mathbf{D}_2)$

$\mathbf{E}'$: no diamond and no club are drawn.

$\mathbf{E}' = (\mathbf{D}_1' \cap \mathbf{C}_2') \cap (\mathbf{C}_1' \cap \mathbf{D}_2') = (\mathbf{D}_1' \cap \mathbf{C}_1') \cap (\mathbf{D}_2' \cap \mathbf{C}_2')$

$$P(\mathbf{E}') = P[(\mathbf{D}_1' \cap \mathbf{C}_1') \cap (\mathbf{D}_2' \cap \mathbf{C}_2')] = P(\mathbf{D}_1' \cap \mathbf{C}_1')P[(\mathbf{D}_2' \cap \mathbf{C}_2') | (\mathbf{D}_1' \cap \mathbf{C}_1')] = (\frac{26}{52})(\frac{25}{51}) = \frac{650}{2652}$$

$$P(\mathbf{E}) = 1 - P(\mathbf{E}') = 1 - \frac{650}{2652} = \frac{2002}{2652}$$

**12.3 - Solved Problem 3:** Two cards are drawn at random without replacement from an ordinary deck of cards.

(a). Find the probability that the first card is a king and the second card is a diamond.

(b). Find the probability that the first card is a king or the second card is a diamond.

**Solutions:**

➤ **(a)**.
$\mathbf{K}_1$: the event that a king is drawn on the first draw.

$\mathbf{D}_1$**:** the event that a diamond is drawn on the first draw.

$\mathbf{D}_2$: the event that a king is drawn on the second draw.

$\mathbf{E} = \mathbf{K}_1 \cap \mathbf{D}_2$: the event that a king is drawn on the first drawing and a diamond on the second drawing.

$\mathbf{K}_1 = (\mathbf{K}_1 \cap \mathbf{D}_1) \cup (\mathbf{K}_1 \cap \mathbf{D}_1')$

$\mathbf{E} = \mathbf{K}_1 \cap \mathbf{D}_2 = [(\mathbf{K}_1 \cap \mathbf{D}_1) \cup (\mathbf{K}_1 \cap \mathbf{D}_1')] \cap \mathbf{D}_2 = [(\mathbf{K}_1 \cap \mathbf{D}_1) \cap \mathbf{D}_2] \cup [(\mathbf{K}_1 \cap \mathbf{D}_1') \cap \mathbf{D}_2]$

$P(\mathbf{E}) = P(\mathbf{K}_1 \cap \mathbf{D}_2) = P([(\mathbf{K}_1 \cap \mathbf{D}_1) \cap \mathbf{D}_2] \cup [(\mathbf{K}_1 \cap \mathbf{D}_1') \cap \mathbf{D}_2]) = P[(\mathbf{K}_1 \cap \mathbf{D}_1) \cap \mathbf{D}_2] + P[(\mathbf{K}_1 \cap \mathbf{D}_1') \cap \mathbf{D}_2] =$

$P(\mathbf{K}_1 \cap \mathbf{D}_1) P[\mathbf{D}_2 | (\mathbf{K}_1 \cap \mathbf{D}_1)] + P(\mathbf{K}_1 \cap \mathbf{D}_1') P[\mathbf{D}_2 | (\mathbf{K}_1 \cap \mathbf{D}_1')] = (\frac{1}{52})(\frac{12}{51}) + (\frac{3}{52})(\frac{13}{51}) = \frac{1}{52}$

➤ **(b)**.
$\mathbf{K}_1$: the event that a king is drawn on the first draw.

$\mathbf{D}_2$: the event that a diamond is drawn on the second draw.

**E:** the event that a king is drawn on the first drawing or a diamond on the second drawing.

$\mathbf{E} = \mathbf{K}_1 \cup \mathbf{D}_2$

$P(\mathbf{D}_2) = 13/52$ (**See 12.3 - Solved Problem 2** ).

$P(\mathbf{E}) = P(\mathbf{K}_1 \cup \mathbf{D}_2) = P(\mathbf{K}_1) + P(\mathbf{D}_2) - P(\mathbf{K}_1 \cap \mathbf{D}_2) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

**12.3-Solved Problem 4:** Three cards are drawn without replacement from an ordinary deck of cards. Let $\mathbf{D}_i$, $\mathbf{C}_i$ be the events that a diamond or club is drawn respectively on the ith drawing.

(a). Show $P(\mathbf{D}_1 \cap \mathbf{C}_3) = P(\mathbf{D}_1 \cap \mathbf{C}_2)$.

(b). Find the probability that at least 2 diamonds are drawn.

(c). Find the probability that at least 1 diamond or 1 club is drawn.

(d). Find the probability that at least 1 diamond and 1 club is drawn.

**Solutions:**

➤ **(a)**.
Step 1: $\mathbf{D}_1 \cap \mathbf{C}_3 = \mathbf{D}_1 \cap (\mathbf{C}_2 \cup \mathbf{C}_2') \cap \mathbf{C}_3 = (\mathbf{D}_1 \cap \mathbf{C}_2 \cap \mathbf{C}_3) \cup (\mathbf{D}_1 \cap \mathbf{C}_2' \cap \mathbf{C}_3)$, (distributive law)

Step 2: $P(\mathbf{D}_1 \cap \mathbf{C}_3) = P(\mathbf{D}_1 \cap \mathbf{C}_2 \cap \mathbf{C}_3) + P(\mathbf{D}_1 \cap \mathbf{C}_2' \cap \mathbf{C}_3) = (13/52)(13/51)(12/50) + (13/52)(38/51)(13/50) = 8450/132600$

Step 3: $P(\mathbf{D}_1 \cap \mathbf{C}_2) = (13/52)(13/51) = (13/52)(13/51)(50/50) = 8450/132600$

➤ **(b)**.
**E**: at least 2 diamonds equals exactly 2 diamonds or 3 diamonds:

Step 1: $\mathbf{E} = \{(\mathbf{D}_1 \cap \mathbf{D}_2 \cap \mathbf{D}_3') \cup (\mathbf{D}_1 \cap \mathbf{D}_2' \cap \mathbf{D}_3) \cup (\mathbf{D}_1' \cap \mathbf{D}_2 \cap \mathbf{D}_3)\} \cup (\mathbf{D}_1 \cap \mathbf{D}_2 \cap \mathbf{D}_3)$, (These sets are disjoint.)

Step 2: $P(\mathbf{E}) = P(\mathbf{D}_1 \cap \mathbf{D}_2 \cap \mathbf{D}_3') + P(\mathbf{D}_1 \cap \mathbf{D}_2' \cap \mathbf{D}_3) + P(\mathbf{D}_1' \cap \mathbf{D}_2 \cap \mathbf{D}_3) + P(\mathbf{D}_1 \cap \mathbf{D}_2 \cap \mathbf{D}_3)$

(13/52)(12/51)(39/50) + (13/52)(39/51)(12/50) + (39/52)(4/51)(3/50) + (13/52)(12/51)(11/50) = 7800/132600

➤ **(c)**.
Step 1: $\mathbf{E} = (\mathbf{D}_1 \cup \mathbf{C}_1) \cup (\mathbf{D}_2 \cup \mathbf{C}_2) \cup (\mathbf{D}_3 \cup \mathbf{C}_3)$

Step 2: $\mathbf{E}' = [(\mathbf{D}_1 \cup \mathbf{C}_1) \cup (\mathbf{D}_2 \cup \mathbf{C}_2) \cup (\mathbf{D}_3 \cup \mathbf{C}_3)]' = (\mathbf{D}_1' \cap \mathbf{C}_1') \cap (\mathbf{D}_2' \cap \mathbf{C}_2') \cap (\mathbf{D}_3' \cap \mathbf{C}_3')$ (DeMorgan law)

Step 3: $\mathbf{E}' = P[(\mathbf{D}_1' \cap \mathbf{C}_1') \cap (\mathbf{D}_2' \cap \mathbf{C}_2') \cap (\mathbf{D}_3' \cap \mathbf{C}_3')]$ = (26/52)(25/51)(24/50) = 15600/132600

Step 4: $P(\mathbf{E}) = 1 - P(\mathbf{E}')$ = 1 - 15600/132600 = 117000/132600

➤ **(d)**.
**E**: The event that at least 1 diamond and 1 club is drawn

$\mathbf{E} = (\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3) \cap (\mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3)$

$\mathbf{E}' = \{(\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3) \cap (\mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3)\}' = (\mathbf{D}_1' \cap \mathbf{D}_2' \cap \mathbf{D}_3') \cup (\mathbf{C}_1' \cap \mathbf{C}_2' \cap \mathbf{C}_3')$, (DeMorgan's law)

$P(\mathbf{E}') = P(\mathbf{D}_1' \cap \mathbf{D}_2' \cap \mathbf{D}_3') + P(\mathbf{C}_1' \cap \mathbf{C}_2' \cap \mathbf{C}_3') - P\{(\mathbf{D}_1' \cap \mathbf{C}_1') \cap (\mathbf{D}_2' \cap \mathbf{C}_2') \cap (\mathbf{D}_3' \cap \mathbf{C}_3')\} =$

(39/52)(38/51)(37/50) + (39/52)(38/51)(37/50) - (26/52)(25/51)(24/50) = 94068/132600

$P(\mathbf{E}) = 1 - P(\mathbf{E})$ = 1 - 94068/132600 = 38532/132600

**12.3 - Solved Problem 5:** Two urns are sitting on a table. Urn 1 has 3 red marbles and 7 white marbles. Urn 2 has 6 red marbles and 3 white marbles. A marble is selected from urn 1 and placed in urn 2. Next a marble is selected from urn 2. Find the probability that the marble selected from urn 2 is red.

**Solution**:

Let $\mathbf{R}_1$ be the event that a red is drawn from urn 1 and placed in urn 2. Let $\mathbf{R}_2$ be the event that a red is drawn from urn 2. Then,

$\mathbf{R}_2 = (\mathbf{R}_1 \cap \mathbf{R}_2) \cup (\mathbf{R}_1' \cap \mathbf{R}_2)$

$P(\mathbf{R}_1) = 3/10$

$P(\mathbf{R}_2 | \mathbf{R}_1) = 7/10$

$P(\mathbf{R}_1') = 7/10$

$P(\mathbf{R}_2|\mathbf{R}_1') = 6/10$

$P(\mathbf{R}_1 \cap \mathbf{R}_2) = P(\mathbf{R}_1)P(\mathbf{R}_2|\mathbf{R}_1) = (3/10)(7/10) = 21/100$

$P(\mathbf{R}_1' \cap \mathbf{R}_2) = P(\mathbf{R}_1')P(\mathbf{R}_2|\mathbf{R}_1') = (7/10)(6/10) = 42/100$

$P(\mathbf{R}_2) = P(\mathbf{R}_1 \cap \mathbf{R}_2) + P(\mathbf{R}_1' \cap \mathbf{R}_2) = 21/100 + 42/100 = 63/100$

**12.3 - Solved Problem 6**: According to the Arizona Chapter of the American Lung Association, 7.0% of the population has lung disease. Of those people that have lung disease, 90% smoke and of those not having lung disease, 25.3% are smokers. Suppose a person is selected at random from the population. Find the probability that the person selected is a smoker.

**Solution:**

Let **S** be the event that the person selected is a smoker.

Let **D** be the event that the person has a lung disease.

From the problem, $P(\mathbf{D})$ is the probability that the person selected has lung disease: $P(\mathbf{D}) = 0.07$.

$P(\mathbf{S}|\mathbf{D})$ is given that the person has a lung disease, the probability the person smokes: $P(\mathbf{S}|\mathbf{D}) = 0.90$.

$P(\mathbf{S}|\mathbf{D}')$ is given that the person does not have a lung disease, the probability the person smokes:

$P(\mathbf{S}|\mathbf{D}') = 0.253$.

Now $\mathbf{S} = (\mathbf{S} \cap \mathbf{D}) \cup (\mathbf{S} \cap \mathbf{D}')$ and

$P(\mathbf{S}) = P(\mathbf{S} \cap \mathbf{D}) + P(\mathbf{S} \cap \mathbf{D}') = P(\mathbf{D})P(\mathbf{S}|\mathbf{D}) + P(\mathbf{D}')P(\mathbf{S}|\mathbf{D}') = (0.070)(0.90) + (0.93)(0.253) \approx 0.30$.

## Unsolved Problems with Answers

**12.3 - Problem 1:** Two urns are sitting on a table. Urn 1 has 5 blue marbles, 10 red marbles and 15 white marbles. Urn 2 has 10 blue marbles, 10 red marbles and 10 white marbles. A card is selected from an ordinary deck of cards. If a king occurs, a marble is selected at random from urn 1; otherwise a marble is selected at random from urn 2. Find the probability that the marble selected is white or red.

**Answer:**

53/78

⇑ *Refer back to* **12.3 - Example 1 & 12.3 - Solved Problem 1.**

**12.3 - Problem 2:** Two cards are drawn at random without replacement from an ordinary deck of cards.

(a). Find the probability that the second card drawn is a face card.

(b). Find the probability that the first card is a face card or the second card is a face card.

(c). Find the probability that a face card and ace are drawn.

(d). Find the probability that at least one face card or one ace is drawn.

**Answers:**

➤ **(a)**. 12/52

➤ **(b)**. 1092/2652

➤ **(c)**. 96/2652

➤ **(d)**. 1392/2652

⇑ *Refer back to* **12.3 - Example 2 & 12.3 - Solved Problem 2**.

**12.3 - Problem 3:** Two cards are drawn at random without rreplacement from an ordinary deck of cards.

(a). Find the probability that the first card is a diamond and the second card is a face card.

(b). Find the probability that the first card is a diamond or the second card is a face card.

.

**Answers:**

➤ **(a)**. 3/52

➤ **(b)**. 22/52

⇑ *Refer back to* **12.3 - Example 3 & 12.3 - Solved Problem 3**.

**12.3- Problem 4:** Three cards are drawn without replacement from an ordinary deck of cards. Let $\mathbf{F}_i$, $\mathbf{A}_i$ be the events that a face card or an ace is drawn respectively on the ith drawing.

(a). Show $P(\mathbf{F}_1 \cap \mathbf{A}_3) = P(\mathbf{F}_1 \cap \mathbf{A}_2)$.

(b). Find the probability that at least 2 face cards are drawn.

(c). Find the probability that at least 1 face card or 1 ace is drawn.

(d). Find the probability that at least 1 face card and 1 ace is drawn.

**Answers:**

➤ **(a)**. $P(\mathbf{F}_1 \cap \mathbf{A}_3) = (12/52)(4/51)(3/50) + (12/52)(47/51)(4/50) = (12/52)(4/51)(50/50) = P(\mathbf{F}_1 \cap \mathbf{A}_2)$

➤ **(b)**. 17160/132600

➤ **(c)**. 89760/132600

➤ **(d)**. 120216/132600

⇑ *Refer back to* **12.3 - Example 4 & 12.3 - Solved Problem 4**.

**12.3 - Problem 5:** Two urns are sitting on a table. Urn 1 has 3 red marbles and 7 white marbles. Urn 2 has 4 blue marbles, and 10 white marbles. A marble is selected from urn 1 and placed in urn 2. Next a marble is selected from urn 2. Find the probability that the marble selected from urn 2 is white.

**Answer:**

107/150

⇑ *Refer back to* **12.3 - Example 5 & 12.3 - Solved Problem 5**.

**12.3 - Problem 6:** Two large toy companies wish to sell talking teddy bears. One company, Toys International estimates there is a 75% chance their teddy bear will make a profit for the company provided the competing toy company does not introduce a talking teddy bear on the market and a 35% chance it will be profitable if the competing company introduces such a toy on the market. Further, it estimates there is a 60% chance the competing company will introduce the toy. Find the probability that the teddy bear is profitable.

**Answer:**

0.51

⇑ *Refer back to* **12.3 - Example 6 & 12.3 - Solved Problem 6**.

# 12.4 - Mutually Independent Events

### Two Independent Events

Two events are said to be independent if the occurrence of one event has no affect on the occurrence of the other event. This can be tested in two ways:

> Two events **A** and **B** are independent if
>
> 1. $P(\mathbf{A}|\mathbf{B}) = P(\mathbf{A})$
>
> or
>
> 2. $P(\mathbf{A}\cap\mathbf{B}) = P(\mathbf{A})P(\mathbf{B})$.

Events that are not independent are said to be dependent.

It can be shown that if **A** and **B** are independent events then $P(\mathbf{A}'|\mathbf{B}) = P(\mathbf{A}')$.

**Pair-wise Independent Events**

> A sequence of events $\mathbf{A}_1, \mathbf{A}_2,..., \mathbf{A}_n$ is said to be pair-wise independent if
>
> $P\{\mathbf{A}_i \cap \mathbf{A}_k\} = P(\mathbf{A}_i)P(\mathbf{A}_k)$ for all j, k $(1 \le j < k \le n)$.

**Mutually Independent Events**

> A sequence of events $\mathbf{A}_1, \mathbf{A}_2,..., \mathbf{A}_n$ is said to be mutually independent if for all combinations $(1 \le j < k <... \le n)$ the multiplication rules
>
> $P\{\mathbf{A}_i \cap \mathbf{A}_k\} = P(\mathbf{A}_i)P(\mathbf{A}_k)$
>
> $P\{\mathbf{A}_i \cap \mathbf{A}_j \cap \mathbf{A}_k\} = P(\mathbf{A}_i)P(\mathbf{A}_j)P(\mathbf{A}_k)$
>
> ::::::::::::::::::::::::::::::::::::::::::::::::::
>
> $P\{\mathbf{A}_1 \cap \mathbf{A}_2 \cap ... \cap \mathbf{A}_n\} = P(\mathbf{A}_1)P(\mathbf{A}_2)...P(\mathbf{A}_n)$ hold.

Although there are examples of pair wise independent sequence of events that are not mutually independent (see supplementary problems 34 and 35), we shall assume, unless otherwise stated, that all independent events are mutually independent.

**12.4 - Example 1:** A card is drawn from an ordinary deck of cards. Show that the event a

(a). king is drawn and the event a diamond is drawn are independent events.

(b). king is drawn and the event queen is drawn are dependent events.

**Solutions:**

➤ **(a)**.
**K**: The event a king is drawn.
**D**: The event a diamond is drawn.
$\mathbf{K} \cap \mathbf{D}$: A king of diamonds is drawn.

$P(\mathbf{K} \cap \mathbf{D}) = 1/52$

$P(\mathbf{K}) = 4/52 = 1/13$

P(**D**) = 13/52 = 1/4

P(**K**)P(**D**) = (1/13)(1/4) = 1/52= P(**K**∩**D**), which shows that the two events are independent.

➤ **(b)**.
**K**: The event a king is drawn.
**Q**: The event a queen is drawn.

P(**Q**|**K**) = 0 ≠ P(**Q**)

Therefore, the two events are dependent.

**12.4 - Example 2:** Two cards are drawn from an ordinary deck of cards without replacement.

(a), Show that the event a face card is drawn on the first drawing and the event a diamond card is drawn on the second drawing are independent.

(b). Show that the event a king is drawn on the first drawing and the event a face card is drawn on the second drawing are dependent.

**Solutions:**

➤ **(a)**.
$\mathbf{F}_1$**:** the event a face card is drawn on the first drawing.

$\mathbf{D}_2$**:** the event a diamond card is drawn on the second drawing.

$P(\mathbf{D}_1) = 13/52 = 1/4$

$P(\mathbf{F}_2) = 12/52$ (**See 12.3 - unsolved problem 2** ).

$\mathbf{D}_1 = (\mathbf{D}_1 \cap \mathbf{F}_2) \cup (\mathbf{D}_1 \cap \mathbf{F}_2')$

$P(\mathbf{D}_1 \cap \mathbf{F}_2) = P\{[(\mathbf{D}_1 \cap \mathbf{F}_1) \cup (\mathbf{D}_1 \cap \mathbf{F}_1')] \cap \mathbf{F}_2\} = P\{[(\mathbf{D}_1 \cap \mathbf{F}_1) \cap \mathbf{F}_2] \cup [(\mathbf{D}_1 \cap \mathbf{F}_1') \cap \mathbf{F}_2]\} =$

$P\{(\mathbf{D}_1 \cap \mathbf{F}_1) \cap \mathbf{F}_2\} + P\{(\mathbf{D}_1 \cap \mathbf{F}_1') \cap \mathbf{F}_2\} = P(\mathbf{D}_1 \cap \mathbf{F}_1)P(\mathbf{F}_2 | \mathbf{D}_1 \cap \mathbf{F}_1) + P(\mathbf{D}_1 \cap \mathbf{F}_1')P(\mathbf{F}_2 | \mathbf{D}_1 \cap \mathbf{F}_1') =$

$(3/52)(11/51) + (10/52)(12/51) = 153/2652 = 3/52$

$P(\mathbf{D}_1)P(\mathbf{F}_2) = (13/52)(12/52) = 3/52$

Therefore, $P(\mathbf{D}_1 \cap \mathbf{F}_2) = P(\mathbf{D}_1 P(\mathbf{F}_2)$.

which shows independence of the two events.

➤ **(b)**.
$\mathbf{K}_1$**:** the event a king is drawn on the first drawing.

$\mathbf{F}_2$: the event a face is drawn on the second drawing.

$P(\mathbf{F}_2|\mathbf{K}_1) = 11/51 \neq P(\mathbf{F}_2) = 12/52$ .

Therefore, the these two events are dependent.

**12.4 - Example 3**: A recent study of criminal arrests based on gender showed the following data:

| | **Males** | **Females** | **Total** |
|---|---|---|---|
| **Guilty** | 150 | 225 | 375 |
| **Innocent** | 50 | 75 | 125 |
| **Total** | 200 | 300 | 500 |

From this group, a person is selected at random. Let **M** be the event a male is selected and **G** the event that the person is guilty. Find out if these two events are independent.

**Solution:**

We will use the test:

If the events **M** and **G** are independent then $P(\mathbf{G}|\mathbf{M}) = P(\mathbf{G})$: Given that a male is selected the probability that he is guilty is the same as the probability that a person (male or female) selected at random is guilty.

Since $\#(\mathbf{G}\cap\mathbf{M}) = 150$, $\#\mathbf{M} = 200$, $\#\mathbf{G} = 375$,

**Step 1:** $$P(\mathbf{G}|\mathbf{M}) = \frac{P(\mathbf{G}\cap\mathbf{M})}{P(\mathbf{M})} = \frac{\frac{150}{500}}{\frac{200}{500}} = \frac{150}{200} = \frac{3}{4}.$$

**Step 2:** $$P(\mathbf{G}) = \frac{375}{500} = \frac{3}{4}.$$

**Step 3:** $$P(\mathbf{G}|\mathbf{M}) = P(\mathbf{G}) = \frac{3}{4}.$$

Therefore, **G** and **M** are independent.

**12.4 - Example 4**: A survey was made of families with three children. One such family is picked at random. Let **B** be the event that the family has both sexes and **G** the event that at most one girl is in the family. Find out if these two events are independent.

**Solution**:

Assume **S** is the sample space in order of birth.

$\mathbf{S}$ = {(g,g,b), (g,b,g), (b,g,g), (g,g,g), (b,b,g), (b,g,b), (g,b,b), (b,b,b)}

**B:** the event that the family has at least one girl and one boy = {(b,b,g), (b,g,b), (g,b,b), (g,g,b), (g,b,g), (b,g,g)}.

**G**: the event that the family has one girl or no girls = {(b,b,b), (b,b,g), (b,g,b), (g,b,b)}.

**B∩G**: the event that the family has exactly one girl and two boys = {(b,b,g), (b,g,b), (g,b,b)}.

To test if the events **B** and **G** are independent we use $P(\mathbf{B}\cap\mathbf{G}) = P(\mathbf{B})P(\mathbf{G})$.

**Step 1:** $P(\mathbf{B}\cap\mathbf{G}) = \dfrac{\#(\mathbf{B}\cap\mathbf{G})}{\#\mathbf{S}} = \dfrac{3}{8}$

**Step 2:** $P(\mathbf{B}) = \dfrac{\#\mathbf{B}}{\#\mathbf{S}} = \dfrac{6}{8}$

**Step 3:** $P(\mathbf{G}) = \dfrac{\#\mathbf{G}}{\#\mathbf{S}} = \dfrac{4}{8}$

**Step 4:** $P(\mathbf{B})P(\mathbf{G}) = (\dfrac{6}{8})(\dfrac{4}{8}) = \dfrac{3}{8}$

Therefore, $P(\mathbf{B}\cap\mathbf{G}) = P(\mathbf{B})P(\mathbf{G})$.

Thus, **B** and **G** are independent.

**12.4 - Example 5:** Mr. Hope is a famous baseball handicapper. He claims that his chance of predicting a winning game is 0.60. Assume he bets on 3 games. Find the probability that he wins 2 games.
**Solution:**

Let $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ represent the events that a win occurs on the first, second, and third games respectively. It is reasonable to assume that these events are mutually independent since one selection should not affect another selection.

**T:** The event that he wins on exactly two games.

The following are the ways he can win exactly two games:

He wins the first and second games but loses the third game: $\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3'$.

He wins the first and third games but loses the second game: $\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3$.

He loses the first game but wins the second and third game: $\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3$.

The event that he wins 2 games is the union of the three above events:

$\mathbf{T} = (\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3')\cup(\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3)\cup(\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3)$.

Since these three events are disjoint we have

$P(\mathbf{T}) = P(\mathbf{W}_1\cap\mathbf{W}_2\cap\mathbf{W}_3') + P(\mathbf{W}_1\cap\mathbf{W}_2'\cap\mathbf{W}_3) + P(\mathbf{W}_1'\cap\mathbf{W}_2\cap\mathbf{W}_3)$

Since $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ are mutually independent events, we have

$P(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3') = P(\mathbf{W}_1)P(\mathbf{W}_2)P(\mathbf{W}_3')$

$P(\mathbf{W}_1 \cap \mathbf{W}_2' \cap \mathbf{W}_3) = P(\mathbf{W}_1)P(\mathbf{W}_2')P(\mathbf{W}_3)$

$P(\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3) = P(\mathbf{W}_1')P(\mathbf{W}_2)P(\mathbf{W}_3)$

We can now write $P(\mathbf{T}) = P(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3') + P(\mathbf{W}_1 \cap \mathbf{W}_2' \cap \mathbf{W}_3) + P(\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3) =$

$P(\mathbf{W}_1)P(\mathbf{W}_2)P(\mathbf{W}_3') + P(\mathbf{W}_1)P(\mathbf{W}_2')P(\mathbf{W}_3) + P(\mathbf{W}_1')P(\mathbf{W}_2)P(\mathbf{W}_3) =$

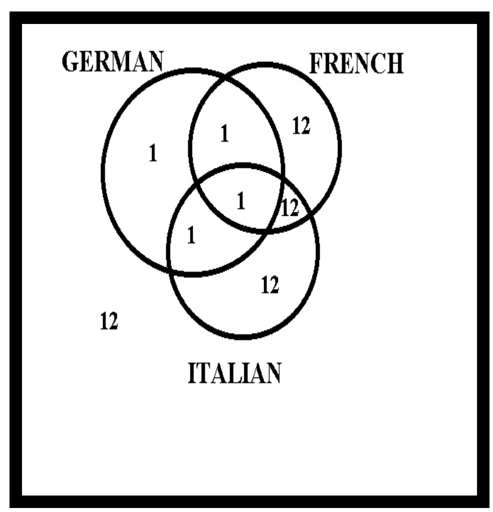
$(0.60)(0.60)(0.4) + (0.6)(0.4)(0.6) + (0.4)(0.6)(0.6) = 3(0.6)^2(0.4) = 0.432.$

**12.4 - Example 6:** At a local college, a survey was taken in a class of 52 foreign language students. The following was the result of this survey:

1 students speaks German(**G**),French(**F**) and Italian (**I**).
2 students speaks French and German.
2 students speaks Italian and German.
13 students speaks Italian and French.
4 students speaks German
26 students speaks French
26 students speaks Italian
A student is randomly selected. Show that the events **G**, **F, I** are mutually independent.

**Solution:**

From the Venn diagram constructed from the data above we find:
$\#\mathbf{G} = 4$
$\#\mathbf{F} = 26$
$\#\mathbf{I} = 26$
$\#\mathbf{S} = 52$, the cardinality of the sample space **S**.




$\#(\mathbf{F} \cap \mathbf{G}) = 2$
$\#(\mathbf{I} \cap \mathbf{G}) = 2$
$\#(\mathbf{F} \cap \mathbf{I}) = 13$
$\#(\mathbf{F} \cap \mathbf{G} \cap \mathbf{I}) = 1$

$$P((\mathbf{F} \cap \mathbf{G}) = \frac{2}{52} = \frac{1}{26}$$

$$P(\mathbf{F})P(\mathbf{G}) = (\frac{26}{52})(\frac{4}{52}) = \frac{1}{26}$$

Therefore, $P(\mathbf{F} \cap \mathbf{G}) = P(\mathbf{F})P(\mathbf{G})$

$P(\mathbf{I}\cap\mathbf{G}) = \frac{2}{52} = \frac{1}{26}$

$P(\mathbf{I})P(\mathbf{G}) = (\frac{26}{52})(\frac{4}{52}) = \frac{1}{26}$

Therefore, $P(\mathbf{I}\cap\mathbf{G}) = P(\mathbf{I})P(\mathbf{G})$

$P(\mathbf{I}\cap\mathbf{F}) = \frac{1}{4}$

$P(\mathbf{I})P(\mathbf{F}) = (\frac{26}{52})(\frac{26}{52}) = \frac{1}{4}$

Therefore, $P(\mathbf{I}\cap\mathbf{F}) = P(\mathbf{I})P(\mathbf{F})$

$P(\mathbf{I}\cap\mathbf{F}\cap\mathbf{G}) = \frac{1}{52}$

$P(\mathbf{I})P(\mathbf{F})P(\mathbf{G}) = (\frac{26}{52})(\frac{26}{52})(\frac{4}{52}) = \frac{1}{52}$

Therefore, $P(\mathbf{I}\cap\mathbf{F}\cap\mathbf{G}) = P(\mathbf{I})P(\mathbf{F})P(\mathbf{G})$

We conclude that the three events are mutually independent.

## Solved Problems

**12.4 - Solved Problem 1:** A card is drawn from an ordinary deck of cards. Show that the event a

(a). face card is drawn and the event a diamond is drawn are independent events.

(b). face card is drawn and the event a king is drawn are dependent events.

**Solutions:**

➤ **(a)**.
**F**: The event a face card is drawn.
**D**: The event a diamond is drawn.
**F∩D**: A face card that is also a diamond is drawn.

$P(\mathbf{F}\cap\mathbf{D}) = 3/52$

$P(\mathbf{F}) = 12/52 = 3/13$

$P(\mathbf{D}) = 13/52 = 1/4$

P(**F**)P(**D**) = (3/13)(1/4) = 3/52 = P(**F**∩**D**), which shows that the two events are independent.

➤ **(b)**.
**F**: The event a face card is drawn.
**K**: The event a king is drawn.
**F**∩**K**: A face card that is also a king is drawn.

P(**F**|**K**) = 1

P(**F**) = 12/52 = 3/13 ≠ 1

which shows that the two events are dependent.

**12.4 - Solved Problem 2:** Two cards are drawn from an ordinary deck of cards without replacement.

(a). Show that the event a diamond is drawn on the first drawing and the event a king is drawn on the second drawing are independent.

(b). Show that the event a king is drawn on the first drawing and the event a king is drawn on the second drawing are dependent.

**Solutions:**

➤ **(a)**.
$\mathbf{D}_1$**:** the event a diamond is drawn on the first drawing.
$\mathbf{K}_2$**:** the event a king is drawn on the second drawing.

$P(\mathbf{D}_1) = 13/52 = 1/4$
$P(\mathbf{K}_2) = 4/52 = 1/13$ (See **3.3 Example 3**.)

$\mathbf{D}_1 = (\mathbf{D}_1 \cap \mathbf{K}_1) \cup (\mathbf{D}_1 \cap \mathbf{K}_1{}')$

$P(\mathbf{D}_1 \cap \mathbf{K}_2) = P\{[(\mathbf{D}_1 \cap \mathbf{K}_1) \cup (\mathbf{D}_1 \cap \mathbf{K}_1{}')] \cap \mathbf{K}_2\} = P\{[(\mathbf{D}_1 \cap \mathbf{K}_1) \cap \mathbf{K}_2] \cup [(\mathbf{D}_1 \cap \mathbf{K}_1{}') \cap \mathbf{K}_2]\} =$

$P\{(\mathbf{D}_1 \cap \mathbf{K}_1) \cap \mathbf{K}_2\} + P\{(\mathbf{D}_1 \cap \mathbf{K}_1{}') \cap \mathbf{K}_2\} = P(\mathbf{D}_1 \cap \mathbf{K}_1)P(\mathbf{K}_2 | \mathbf{D}_1 \cap \mathbf{K}_1) + P(\mathbf{D}_1 \cap \mathbf{K}_1{}')P(\mathbf{K}_2 | \mathbf{D}_1 \cap \mathbf{K}_1{}') =$

$$\left(\frac{1}{52}\right)\left(\frac{3}{51}\right) + \left(\frac{12}{52}\right)\left(\frac{4}{51}\right) = \frac{1}{52} = \left(\frac{1}{4}\right)\left(\frac{1}{13}\right) = P(\mathbf{D}_1)P(\mathbf{K}_2),$$

which shows independence of the two events.

➤ **(b)**.
$\mathbf{K}_1$**:** the event a king is drawn on the first drawing.
$\mathbf{K}_2$**:** the event a king is drawn on the second drawing.

$P(\mathbf{K}_2 | \mathbf{K}_1) = 3/51 \neq P(\mathbf{K}_2) = 4/52$ (See **3.3 - Example 3**.)

**12.4 - Solved Problem 3:** A recent study of criminal arrests based on gender showed the following data:

| | **Males** | **Females** | **Total** |
|---|---|---|---|
| **Guilty** | 150 | 225 | 375 |
| **Innocent** | 50 | 75 | 125 |
| **Total** | 200 | 300 | 500 |

From this group, a person is selected at random. Let **F** be the event a female is selected and **I** the event that the person is innocent. Find out if these two events are independent.

**Solution:**

Since $\#(\mathbf{I}\cap\mathbf{F}) = 75$, $\#\mathbf{F} = 300$, $\#\mathbf{I} = 125$,

**Step 1:** $P(\mathbf{I}|\mathbf{F}) = \dfrac{P(\mathbf{I}\cap\mathbf{F})}{P(\mathbf{F})} = \dfrac{\frac{75}{500}}{\frac{300}{500}} = \dfrac{75}{300} = \dfrac{1}{4}$

**Step 2:** $P(\mathbf{I}) = \dfrac{125}{500}$

Therefore, **F** and **I** are independent.

**12.4 - Solved Problem 4:** A survey was made of families with three children. One such family is picked at random. Let **B** be the event that the family has both sexes and **G** the event that at least two girls are in the family. Find out if these two events are independent. **S** is the sample space in order of birth.

**Solution:**

**S** = {(g,g,b), (g,b,g), (b,g,g), (g,g,g), (b,b,g), (b,g,b), (g,b,b), (b,b,b)}

**B** = {(b,b,g), (b,g,b), (g,b,b), (g,g,b), (g,b,g), (b,g,g)}.

**G** = {(g,g,g), (g,g,b), (g,b,g), (b,g,g)}

**B∩G =** {(g,g,b), (g,b,g), (b,g,g)}.

**Step 1:** $P(\mathbf{B}\cap\mathbf{G}) = \dfrac{3}{8}$

**Step 2:** $P(\mathbf{B}) = \dfrac{\#\mathbf{B}}{\#\mathbf{S}} = \dfrac{6}{8}$

**Step 3:** $P(\mathbf{B}) = \dfrac{\#\mathbf{G}}{\#\mathbf{S}} = \dfrac{4}{8}$

**Step 4:** P(**B**)P(**G**) = $(\frac{6}{8})(\frac{4}{8}) = \frac{3}{8}$

Thus **B** and **G** are independent.

**12.4 - Solved Problem 5:** Mr. Hope is a famous baseball handicapper. He claims that his chance of predicting a winning game is 0.60. Assume he bets on 3 games. Find the probability that he wins at least 2 games.

**Solution:**

Let $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ represent the events that a win occurs on the first, second, and third games respectively. It is reasonable to assume that these events are mutually independent. Let **T** be the event that he wins at least two games:

$\mathbf{T} = (\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3') \cup (\mathbf{W}_1 \cap \mathbf{W}_2' \cap \mathbf{W}_3) \cup (\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3) \cup (\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3)$

$P(\mathbf{T}) = P(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3') + P(\mathbf{W}_1 \cap \mathbf{W}_2' \cap \mathbf{W}_3) + P(\mathbf{W}_1' \cap \mathbf{W}_2 \cap \mathbf{W}_3) + P(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) =$

$P(\mathbf{W}_1)P(\mathbf{W}_2)P(\mathbf{W}_3') + P(\mathbf{W}_1)P(\mathbf{W}_2')P(\mathbf{W}_3) + P(\mathbf{W}_1')P(\mathbf{W}_2)P(\mathbf{W}_3) + P(\mathbf{W}_1)P(\mathbf{W}_2)P(\mathbf{W}_3) =$

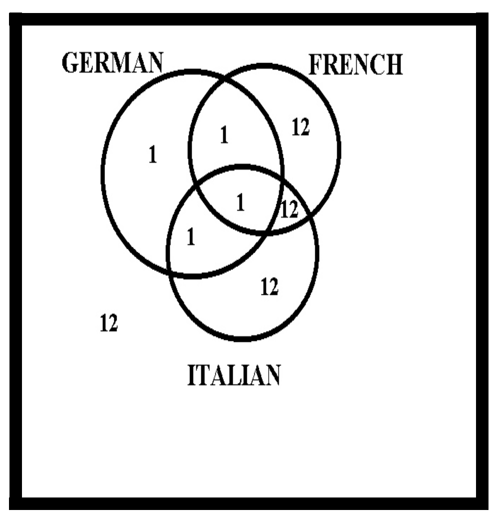
$(0.6)(0.6)(0.4) + (0.6)(0.4)(0.6) + (0.4)(0.6)(0.6) + (0.6)(0.6)(0.6) = 0.144 + 0.144 + 0.144 + 0.216 = 0.648.$

**12.4 - Solved Problem 6:** From 12.4 - Example 5, show that the events **G**′, **F**′, **I**′ are mutually independent.

**Solution:**

From the Venn diagram constructed from the data in Example 4.5, we find:

#**G**′ = 48
#**F**′ = 26
#**I**′ = 26
#**S** = 52, the cardinality of the sample space **S**.




$\#(\mathbf{F}' \cap \mathbf{G}') = \#(\mathbf{F} \cup \mathbf{G})' = 24$

$\#(\mathbf{I}' \cap \mathbf{G}') = \#(\mathbf{I} \cup \mathbf{G})' = 24$

$\#(\mathbf{F}' \cap \mathbf{I}') = \#(\mathbf{F} \cup \mathbf{I})' = 13$

$\#(\mathbf{F}' \cap \mathbf{G}' \cap \mathbf{I}') = \#(\mathbf{F} \cup \mathbf{G} \cup \mathbf{I})' = 12$

$P(\mathbf{F}' \cap \mathbf{G}') = \frac{24}{52} = \frac{6}{13}$

$P(\mathbf{F}')P(\mathbf{G}') = (\frac{26}{52})(\frac{48}{52}) = \frac{6}{13}$

Therefore, $P(\mathbf{F}' \cap \mathbf{G}') = P(\mathbf{F}')P(\mathbf{G}')$.

$$P(\mathbf{I}'\cap\mathbf{G}') = \frac{24}{52} = \frac{6}{13}$$

$$P(\mathbf{I}')P(\mathbf{G}') = (\frac{26}{52})(\frac{48}{52}) = \frac{6}{13}$$

Therefore, $P(\mathbf{I}'\cap\mathbf{G}') = P(\mathbf{I}')P(\mathbf{G}')$.

$$P(\mathbf{I}'\cap\mathbf{F}') = \frac{13}{52} = \frac{1}{4}$$

$$P(\mathbf{I}')P(\mathbf{F}') = (\frac{26}{52})(\frac{26}{52}) = \frac{1}{4}$$

Therefore, $P(\mathbf{I}'\cap\mathbf{F}') = P(\mathbf{I}')P(\mathbf{F}')$.

$$P(\mathbf{I}'\cap\mathbf{F}'\cap\mathbf{G}') = \frac{12}{52} = \frac{3}{13}$$

$$P(\mathbf{I}')P(\mathbf{F}')P(\mathbf{G}') = (\frac{26}{52})(\frac{26}{52})(\frac{48}{52}) = \frac{3}{13}$$

Therefore, $P(\mathbf{I}'\cap\mathbf{F}'\cap\mathbf{G}') = P(\mathbf{I}')P(\mathbf{F}')P(\mathbf{G}')$.

We conclude that the three events are mutually independent.

## Unsolved Problems with Answers

**12.4 - Problem 1:** A card is drawn from an ordinary deck of cards. Show that the event

(a). a king is not drawn and the event a diamond is drawn are independent events.

(b). a king is not drawn and a queen is drawn are dependent events.

**Answers:**

➤ **(a)**.
$P(\mathbf{K}')P(\mathbf{D}) = 3/13 = P(\mathbf{K}'\cap\mathbf{D})$

➤ **(b)**.
$P(\mathbf{K}'|\mathbf{Q}) = 1 \neq P(\mathbf{K}')$

⇑ *Refer back to* **12.4 - Example 1 & 12.4 - Solved Problem 1**.

**12.4 - Problem 2:** Two cards are drawn from an ordinary deck of cards without replacement. Show that the event a

(a). face card is drawn on the first drawing and the event a diamond is not drawn on the second drawing are independent.

(b). king is drawn on the first drawing and the event a queen is not drawn on the second drawing are dependent

**Answers:**

➤ **(a)**.
$P(\mathbf{F}_1)P(\mathbf{D}_2') = 9/52 = P(\mathbf{F}_1 \cap \mathbf{D}_2')$

➤ **(b)**.
$P(\mathbf{Q}_2'|\mathbf{K}_1) = 47/51 \neq P(\mathbf{Q}_2') = 48/52$

⇑ *Refer back to* **12.4 - Example 2& 12.4 - Solved Problem 2**

**12.4 - Problem 3:** In September, 1988, the House of Representatives voted on an amendment requiring life imprisonment for drug-related murders. Results of the vote were reported as shown below:

| | YEA | NAY | DID NOT VOTE | Total |
|---|---|---|---|---|
| **Democrat** | 153 | 83 | 19 | 255 |
| **Republican** | 169 | 0 | 8 | 177 |
| **Total** | 322 | 83 | 27 | 432 |

One person from this group is selected at random. Let **D** be the event that he or she is a Democrat and **Y** the event that this person selected voted for the amendment. Find out if these two events are independent.

**Answer:**

Not independent.

⇑ *Refer back to* **12.4 - Example 2 & 12.4 - Solved Problem 2**.

**12.4 - Problem 4:** A survey was made of families with two children. One such family is picked at random. Let **B** be the event that the family has both sexes and **G** the event that exactly 2 girls are in the family. Find out if these two events are independent.

**Answer:**

Not independent.

⇑ *Refer back to* **12.4 - Example 3 & 12.4 - Solved Problem 3**.

**12.4 - Problem 5:** Mr. Hope is a famous baseball handicapper. He claims that his chance of predicting a winning game is 0.60. Assume he bets on 3 games. Find the probability that he wins only two games in a row.

**Answer:**

0.288

⥣ *Refer back to* **12.4 - Example 4 & 12.4 - Solved Problem 4**.

**12.4 - Problem 6:** From 12.4 - Example 5, show that the events **G**, **F**′, **I**′ are mutually independent.

**Answers:**

$P(\mathbf{G}) = \frac{1}{13}$ , $P(\mathbf{F}') = \frac{1}{2}$ , $P(\mathbf{I}') = \frac{1}{2}$ , $P(\mathbf{G}\cap\mathbf{F}') = \frac{1}{26}$ , $P(\mathbf{G}\cap\mathbf{I}') = \frac{1}{26}$ , $P(\mathbf{I}'\cap\mathbf{F}') = \frac{1}{4}$ ,

$P(\mathbf{G}\cap\mathbf{I}') = \frac{1}{26} = P(\mathbf{G})P(\mathbf{I}') = (\frac{1}{13})(\frac{1}{2})$ , $P(\mathbf{G}\cap\mathbf{F}') = \frac{1}{26} = P(\mathbf{G})P(\mathbf{F}') = (\frac{1}{13})(\frac{1}{2})$ ,

$P(\mathbf{I}'\cap\mathbf{F}') = \frac{1}{4} = P(\mathbf{I}')P(\mathbf{F}') = (\frac{1}{2})(\frac{1}{2})$ , $P(\mathbf{G}\cap\mathbf{I}'\cap\mathbf{F}') = \frac{1}{52} = P(\mathbf{G})P(\mathbf{I}')P(\mathbf{F}') = (\frac{1}{13})(\frac{1}{2})(\frac{1}{2})$

⥣ *Refer back to* **12.4 - Example 5 & 12.4 - Solved Problem 5**.

## Supplementary Problems

1. Two cards are drawn without replacement from an ordinary deck of cards. Find the probability a king and queen are drawn.

2. Two cards are drawn without replacement from an ordinary deck of cards. Find the probability that the first card is a king and the second card is a diamond.

3. An urn contains 5 white marbles, 10 blue marbles and 25 red marbles. Assume three marbles are selected from the urn without replacement. Find the probability that all three marbles are the same color.

4. Assume two cards are drawn from an ordinary deck of cards without replacement. Find the probability that the first card is a king or queen and the second card is a king or queen.
(Hint: $(\mathbf{A}\cup\mathbf{B})\cap(\mathbf{C}\cup\mathbf{D}) = (\mathbf{A}\cap\mathbf{C})\cup(\mathbf{A}\cap\mathbf{D})\cup(\mathbf{B}\cap\mathbf{C})\cup(\mathbf{B}\cap\mathbf{D})$)

5. Assume two cards are drawn from an ordinary deck of cards without replacement. Find the probability that a king or queen are drawn .

6. Mr. Jones received a box containing 100 computer chips, of which five, unknown to him are defective. He decides to select three chips and test them. If at least one chip is defective, he will return the box.

a. Find the probability that he will not return the box.

b. Find the probability that at least two of the three chips selected are defective.

7. Three urns are sitting on a table. Urn 1 has three blue marbles and 7 white marbles, urn 2 has 5 red marbles and 10 white marbles and urn 3 has 10 red marbles and 10 white marbles. A die is tossed once. If a 1 occurs, a marble is selected at random from urn 1; if a 2 or 3 occurs, a marble is selected at random from urn 2 and if

a 4, 5, or 6 occurs, a marble is selected at random from urn 3. Find the probability that the marble selected is white.

8. Assume two cards are drawn from an ordinary deck of cards without replacement. Find the probability that the first card is a king or diamond and the second card is a king.

9. Mr. Hope is a famous baseball handicapper. He claims that his chance of predicting a winning game is 0.60. Assume he bets until he wins.

a. Find the probability that he will stop betting by the third game.

b. Find the probability that he will stop betting on the third game.

10**.** Three urns are sitting on a table. Urn 1 has three blue marbles and 7 white marbles. Urn 2 has 4 blue marbles, and 10 white marbles and urn 3 has 5 white marbles and 5 blue marbles. A marble is selected from urn 1 and placed in urn 2. Next a marble is selected from urn 2 and placed in urn 3. Finally a marble is selected from urn 3. Find the probability that the marble selected from urn 3 is blue.

11. Two decks of ordinary cards are sitting on a table. A card is randomly selected from the first deck and placed in the second deck. Next a card is selected at random from the second deck and placed in the first deck. Finally a card is randomly selected from the first deck. Find the probability that the card finally selected is a diamond.

12. Two urns are sitting on a table. Urn 1 has 5 red and 5 black marbles and Urn 2 has 7 red marbles and 2 black marbles. A die is tossed once. If a 3 appears a marble is drawn at random from urn 1 and placed in urn 2. Next a marble is selected at random from urn 2. Otherwise a marble is drawn at random from urn 2 and placed in urn 1. Next a marble is selected at random from urn 1. Find the probability that the last marble selected is black.

13. A English class has 60 women and 40 men. Students are selected at random one at a time , without replacement, until two males are selected. Find the probability that this process will stop on the fourth selection.

14. Two non-empty events **A** and **B** are said to be mutually exclusive if $\mathbf{A} \cap \mathbf{B} = \phi$. Are such events always dependent?

15. Three cards are drawn from an ordinary deck of cards without replacement. Let $\mathbf{K}_1$ = the event a king is drawn on the first drawing, $\mathbf{K}_2$ = the event a king is drawn on the second drawing and $\mathbf{K}_3$ = the event a king is drawn on the third drawing.

a. Find $P(\mathbf{K}_1 | \mathbf{K}_3)$.

b. Does $P(\mathbf{K}_1 | \mathbf{K}_3) = P(\mathbf{K}_3 | \mathbf{K}_1)$ ?

16. Assume that **A** and **B** are independent events. Show that

a. **A** and $\mathbf{B}'$ are independent. (First show $P(\mathbf{B}' | \mathbf{A}) = 1 - P(\mathbf{B} | \mathbf{A})$).

b. $\mathbf{A}'$ and $\mathbf{B}'$ are independent.

17. Two urns are sitting on a table. Urn A has 6 white marbles and 4 black marbles. Urn B has 3 white marbles

and 7 black marbles. Two marbles are to be drawn from one or both urns under the following rule:

A die is tossed. If a 1 or 2 occurs then 2 marbles are selected from urn A, if a 3, 4, or 5 occurs then 2 marbles are selected from urn B, and if a 6 occurs then one marble is selected from each urn.

Find the probability that both marbles selected are white.

18. Show $P(\mathbf{E}|\mathbf{A}\cup\mathbf{B}) = \dfrac{P(\mathbf{E}\cap\mathbf{A}) + P(\mathbf{E}\cap\mathbf{B}) - P(\mathbf{E}\cap\mathbf{A}\cap\mathbf{B})}{P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A}\cap\mathbf{B})}$ .

19. Show Rule 2: $P(\mathbf{A}\cup\mathbf{B}|\mathbf{E}) = P(\mathbf{A}|\mathbf{E}) + P(\mathbf{B}|\mathbf{E}) - P(\mathbf{A}\cap\mathbf{B}|\mathbf{E})$.

20. Show Rule 1: $P(\mathbf{E}'|\mathbf{B}) = 1 - P(\mathbf{E}|\mathbf{B})$.

21. Two cards are drawn, without replacement, from an ordinary deck of cards. Let $\mathbf{K}_1$ be the event that a king is drawn on the first card and $\mathbf{K}_2$ the event that a king is drawn on the second card. Show that

$P(\mathbf{K}_2|\mathbf{K}_1') \neq 1 - P(\mathbf{K}_2|\mathbf{K}_1)$.

22. Two cards are drawn, without replacement, from an ordinary deck of cards. Let $\mathbf{K}_1$ be the event a king is drawn on the first card, $\mathbf{Q}_1$ the event a queen is drawn on the first card, and $\mathbf{J}_2$ the event that a jack is drawn on the second card. Show that

$P(\mathbf{J}_2|\ \mathbf{K}_1\cup\mathbf{Q}_1) \neq P(\mathbf{J}_2|\mathbf{K}_1) + P(\mathbf{J}_2|\mathbf{Q}_1)$.

Assume an experiment generates a sample space **S** with non-empty events **A** and **B** and **A**, **B** ≠ **S**. For problems 23 - 29 select the correct answers.

23. **A** and **S** are independent. (a). true (b). false (c). undetermined.

24. **A** and **A**′ are independent. (a). true (b). false (c). undetermined.

25. If $\mathbf{A}\cap\mathbf{B} = \phi$ then **A** and **B** are independent. (a). true (b). false (c). undetermined.

26. If $\mathbf{A}\subset\mathbf{B}$ then **A** and **B** are independent. (a). true (b). false (c). undetermined.

27. $\mathbf{A}\cap\mathbf{B} \neq \phi$ then **A** and **B** are independent. (a). true (b). false (c). undetermined.

28. If $P(\mathbf{A}|\mathbf{B}) = P(\mathbf{B}|\mathbf{A})$ then $P(\mathbf{A}) = P(\mathbf{B})$. (a). true (b). false (c). undetermined.

29. If $P(\mathbf{A}) = P(\mathbf{B})$ then $P(\mathbf{A}|\mathbf{B}) = P(\mathbf{B}|\mathbf{A})$ (a). true (b). false (c). undetermined.

30. Show that $P(\mathbf{A}\cap\mathbf{B}\cap\mathbf{C}) = P(\mathbf{A})P(\mathbf{B}|\mathbf{A})P(\mathbf{C}|\mathbf{A}\cap\mathbf{B})$ is always true. (Assume $\mathbf{A}\cap\mathbf{B}\cap\mathbf{C} \neq \phi$).

31. Two cards are drawn from an ordinary deck of cards without replacement. Find the probability that a diamond is drawn on the first drawing and a face card drawn on the second drawing.

32. Assume two cards are drawn, without replacement, from an ordinary deck of cards. If the first card drawn is a king, find the probability that the second card is a diamond.

33. Show that $P(\mathbf{E}_1 \cap \mathbf{E}_2 \cap \mathbf{E}_3 ... \cap \mathbf{E}_n) = P(\mathbf{E}_1)P(\mathbf{E}_2|\mathbf{E}_1)P(\mathbf{E}_3|\mathbf{E}_1 \cap \mathbf{E}_2)P(\mathbf{E}_4|\mathbf{E}_1 \cap \mathbf{E}_2 \cap \mathbf{E}_3)...P(\mathbf{E}_n|\mathbf{E}_1 \cap \mathbf{E}_2 \cap \mathbf{E}_3 ... \cap \mathbf{E}_{n-1})$

34. Assume a die is tossed twice. Let $\mathbf{E}_1$ be the event that an even number occurs on the first toss, $\mathbf{E}_2$ be the event that an even number occurs on the second toss, and **B** the event that the sum of the two numbers is odd.

Show that these events are pair-wise independent but not mutually independent.

35. Assume a die is tossed three times. Let $\mathbf{E}_{1,2}$ be the event that the first and second toss results in the same number, $\mathbf{E}_{1,3}$ be the event that the first and third toss results in the same number, and $\mathbf{E}_{2,3}$ be the event that the second and third toss results in the same number. Show that these events are pair-wise independent but not mutually independent.

36. Assume the events **A**, **B**, **C**, **D** are mutually independent. Show that the two events

a. $\mathbf{E} = \mathbf{A} \cap \mathbf{B}$ and $\mathbf{F} = \mathbf{C} \cap \mathbf{D}$ are independent.

b. $\mathbf{E} = \mathbf{B} \cup \mathbf{C}$ and **D** are independent.

37. Assume an experiment results in events $\mathbf{A}_k$ (k = 1, 2, 3). Using the notation $\mathbf{B}_k = \mathbf{A}_k$ or $\mathbf{A}_k'$, show that the events $\mathbf{A}_k$ are mutually independent if $P(\mathbf{B}_1 \cap \mathbf{B}_2 \cap \mathbf{B}_3) = P(\mathbf{B}_1)P(\mathbf{B}_2)P(\mathbf{B}_3)$. Note: This result can be generalized to any number of events.

38. Cards are drawn from an ordinary deck, without replacement, until 2 diamonds are drawn or 5 cards are drawn whichever occurs first. Find the probability that 5 cards were drawn.

39. Assume 2 cards are drawn without replacement from an ordinary deck of cards. If a king is not drawn on the first drawing, find the probability that a face card is drawn on the second drawing.

40. From Example 4.1, we know that if 1 card is drawn from an ordinary deck of cards, the events “a king is drawn” and the event “a diamond is drawn” are independent events. For these two events show that for any third non -trivial event the three events would be no be mutually independent.

41. A fair coin is tossed until 2 heads occur or 4 tosses, which ever occurs first.

a. Find the probability of each elementary sample value of the sample space **S**.

b. Show that the events that a head occurs on the first toss and a head occurs on the fourth toss are NOT independent events.

c. Find the probability that 4 tosses occurred.

d. If 4 tosses occur, find the probability that a head occurred on the first toss.

42. Two cards are drawn at random without replacement from an ordinary deck of cards.

a. Find the probability that one card is a king and the other card is any kind of diamond..

b. Find the number of ways that one card is a king and the other card is any kind of diamond..

43. Three cards are selected from an ordinary deck of cards. Show that P($\mathbf{K}_1 \cap \mathbf{K}_2$) = P($\mathbf{K}_2 \cap \mathbf{K}_3$) = **P(**$\mathbf{K}_1 \cap \mathbf{K}_3$).

44. In a game of poker, Mr. Jones selects 5 cards from an ordinary deck of cards.

a. In ordinary English, express the event $\mathbf{E} = (\mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_3) \cap (\mathbf{K}_4{}' \cap \mathbf{K}_5{}')$.

b. Find P(**E**).

c. Find the number of ways **E** can happen.

45. From an ordinary deck of cards, 2 hands are placed on the table where each hand has 2 cards.

a. Find the probability that both hands have exactly 1 king.

b. Find the probability that at least 1 hand has exactly 1 king.

46. A fair die is tossed twice.

a. If the sum of the two numbers is 6, find the probability that the first toss resulted in a 2.

b. If the sum of the two numbers is a 5 or 6, find the probability that the first toss resulted in a 2.

47. Two urns contain red and white marbles. Urn A has 6 red and 3 white and urn B has 8 red and 11 white. A card is drawn from an ordinary deck of cards. If the card is a diamond, a marble is drawn from urn A and placed in urn B; then a marble is drawn from urn B. If the card is not a diamond, a marble is drawn from urn B and placed in urn A; then a marble is drawn from urn A. Find the probability that the second marble drawn is red.

*Definition of conditional independence of events:* Assume we have events **A**, **B**, **C**. We define the events **A** and **B** to be conditionally independent relative to **C** if P(**A**∩**B**|**C**) = P(**A**|**C**)P(**B** |**C**). (Assuming P(**C**) > 0.)

48. Two urns are sitting on a table. Urn A has 5 red marbles and 5 black marbles. Urn B has 3 red marbles and 7 black marbles. A fair coin is tossed once. If heads appears, then a marble is first selected from urn A and a second marble is selected from urn B. If tails appears , then a marble is first selected from urn B and a second marble is selected from urn A. Let $\mathbf{R}_1$ be the event that the first marble selected is red and $\mathbf{B}_2$ the event that the second marble selected is black. Let **H** be the event that heads is tossed.

a. Show that the events $\mathbf{R}_1$ and $\mathbf{B}_2$ are not independent.

b. Show that the events $\mathbf{R}_1$ and $\mathbf{B}_2$ are conditionally independent relative to **H**.

49. Assume a fair die is tossed twice. Let **A** be the event that the number 2 appears on the first toss, **B** the event that the number 4 appears on the second toss and **C** the event that the sum of the two numbers is 6.

a. Show that the events **A** and **B** are independent.

b. Show that the events **A**, **B** , and **C** are not mutually independent.

c. Show that the events **A** and **B** are not conditionally independent relative to **C**.

50. Assume a fair die is tossed twice. Let $\mathbf{T}_1$ be the event that the number 2 appears on the first toss, $\mathbf{T}_2$ the event that the number 2 appears on the second toss, $\mathbf{E}_1$ the event that an even number appears on the first toss, and $\mathbf{E}_2$ the event that an even number appears on the second toss.

a. Show that the events $\mathbf{T}_1$ and $\mathbf{T}_2$ are independent.

b. Show that the events $\mathbf{T}_1$ and $\mathbf{T}_2$ are conditionally independent relative to $\mathbf{E}_1 \cap \mathbf{E}_2$.

51. Assume the pair of events **A** , **B** are independent and the pair of events **C**,**D** are also independent. If $\mathbf{A} \subseteq \mathbf{C}$ and $\mathbf{B} \subseteq \mathbf{D}$, show $P(\mathbf{A} \cap \mathbf{B} \mid \mathbf{C} \cap \mathbf{D}) = P(\mathbf{A} \mid \mathbf{C})P(\mathbf{B} \mid \mathbf{D})$.

52. Assume the events **A**, **B** are conditionally independent relative to **C**

a. Show $\mathbf{A}'$ and **B** are conditionally independent relative to **C**.

b. Show $\mathbf{A}'$ and $\mathbf{B}'$ are conditionally independent relative to **C.**

53. A fair die is rolled 3 times. Let $\mathbf{A}_{i,k}$ be the event that the ith and kth rolls produce the same number. Show that the events $\mathbf{A}_{i,k}$ $(1 \le i < k \le 3)$ are pair wise independent but not mutually independent.

54. Three cards are drawn, without replacement, from an ordinary deck of cards. Find the probability that the first card drawn is a face card, the second is a diamond and the third card drawn is a king.

55. Three urns sit on a table. urn 1 has 3 red marbles and 7 black, urn 2 has 12 red and 8 black and urn 3 has 5 red and 5 black. An urn is selected at random and 2 marbles are drawn without replacement from the selected urn. If the first marble selected is red, find the probability that the second marble selected is black.

56. A room contains 2 tables. On each table sits 2 urns containing red and white marbles. The following table contains the contents of the urns.

| **TABLE 1** | | **TABLE 2** | |
|---|---|---|---|
| **URN 1** | **URN 2** | **URN 3** | **URN 4** |
| 12 red marbles<br>8 white marbles | 15 red marbles<br>5 white marbles | 10 red marbles<br>10 white marbles | 5 red marbles<br>15 red marbles |

A table and an urn sitting on the table are selected at random. From the urn, 1 marble is selected. Find the probability the marble selected is red.

57. Assume 3 urns sit on a table. Urn A has 3 red and 7 white marbles; urn B has 7 red and 3 white marbles; and urn C has 5 red and 5 black marbles. Two urns are selected at random and from each urn 1 marble is drawn. Assume the first marble drawn is selected according to the order of the alphabet of the urn's label. (For example, if urns B and C were selected then the first marble would be drawn from urn B and the second from urn C). Show that the event that both marbles selected are red is not independent.

58. Two urns each contain marbles. Urn A contains 3 red marbles and 5 white marbles and urn B contains 7 red marbles and 3 white marbles. A fair die is tossed once. If the number 3 appears, a marble is selected from Urn A otherwise a marble is selected from Urn B. If a red marble is selected, what is the probability that Urn A was selected?