Statistical Inference Theory
Lesson 34 (Optional)*
Combining Type I & Type II Errors

When designing a one-sided test, the Type I and Type II errors can first be decided upon before sampling is actually carried out. The following examples and problems are a continuation of those in Lesson 33.

34.1-Creating decision rules from α and β for one-sided tests

34.1 - Example 1: In order to attract more winter tourists, a southern Florida resort hotel purchased a magazine advertisement that circulates in the New York city area. Part of the advertisement claimed that the average temperature in the resort area during the month of January is $\mu = 75$ degrees Fahrenheit. To challenge this claim, N past years are selected at random. Assume a standard deviation of 7^o.

(a). State H_o and H_a.

(b). For the following decision rule,

D.R.: Take a sample of size N. If $\overline{X} < c^*$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

Find N and c* so that $\alpha = 0.05$ and $\beta = 0.05$ when $\mu = 73^o$.

(c). Restate the decision rule.

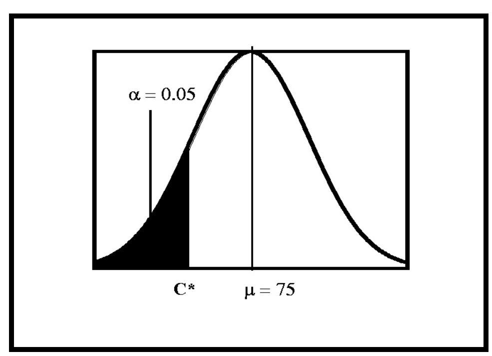
Solutions:

➤**(a)**.
H_o: $\mu = 75^o$

H_a: $\mu < 75^o$

➤**(b)**.
<u>Case 1: Type I error</u>

Step 1: $\mu = 75$



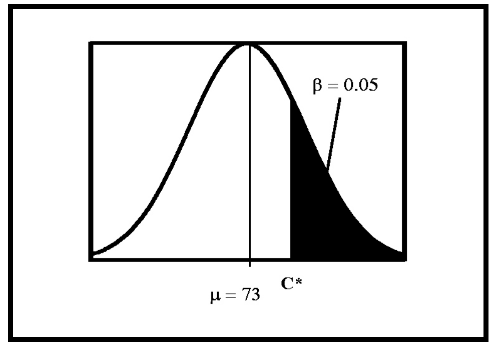
* *This lesson is not required for the understanding of the sequel and can be omitted.*

$$c^* = \mu + z\sigma_{\bar{x}} = 75 + z\frac{\sigma}{\sqrt{N}} = 75 + z\frac{7}{\sqrt{N}}$$

Step 2: $P\{\bar{X} < c^* < 75\} = 0.05$

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45:
z = - 1.64 .

$$c^* = 75 + z\frac{7}{\sqrt{N}} = 75 - (1.64)\frac{7}{\sqrt{N}} = 75 - \frac{11.48}{\sqrt{N}}$$

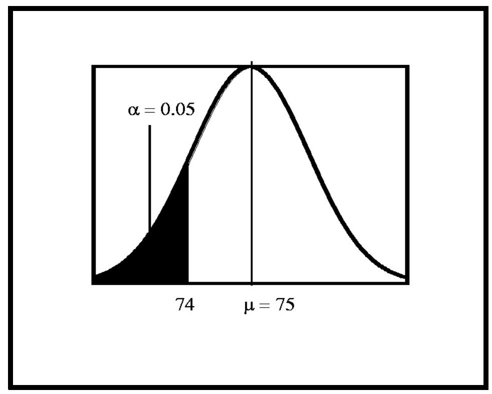


$$c^* = 75 - \frac{11.48}{\sqrt{N}}$$, equation 1

<u>Case 2: Type II error</u>

Step 1: $\mu = 73$

$$c^* = \mu + z\sigma_{\bar{x}} = 73 + z\frac{\sigma}{\sqrt{N}} = 73 + z\frac{7}{\sqrt{N}}$$



Step 2: $P\{73 < c^* < \bar{X}\} = 0.05$

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45: z = 1.64 .

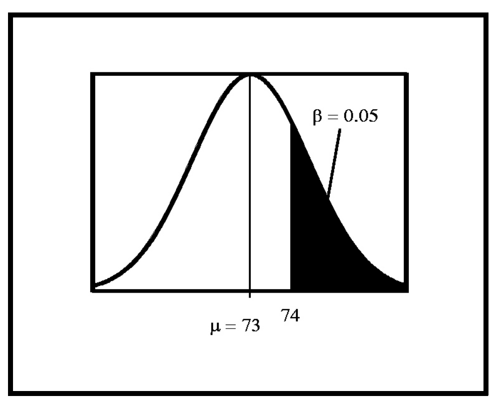
$$c^* = 73 + z\frac{7}{\sqrt{N}} = 73 + (1.64)\frac{7}{\sqrt{N}} = 73 + \frac{11.48}{\sqrt{N}}$$

$$c^* = 73 + \frac{11.48}{\sqrt{N}}$$, equation 2

Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$$75 - \frac{11.48}{\sqrt{N}} = 73 + \frac{11.48}{\sqrt{N}}$$

$$2 = \frac{22.96}{\sqrt{N}}$$



$$\sqrt{N} = \frac{22.96}{2} \approx 11$$

$$N = 121$$

From equation 2,

$$c^* = 73 + \frac{11.48}{\sqrt{N}} = 73 + \frac{11.48}{(\frac{22.96}{2})} = 74$$

➤**(c)**.
D.R.: Take a sample of size 121. If $\overline{X} < 74$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

34.1 - Example 2: The manufacturer of fluorescent light bulbs advertises that the average life of a light bulb is $\mu = 15{,}000$ hours of burning. A consumer group doubts this claim. Assume a random sample of N bulbs is selected and a standard deviation of 700 hours.

(a). State H_o and H_a.

(b). For the following decision rule,

D. R.: Take a sample of size N. If $\overline{X} < c^*$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

Find N and c* so that $\alpha = 0.05$ and $\beta = 0.01$ when $\mu = 14{,}700$.

(c). Restate the decision rule.

Solutions:

➤**(a)**.
H_o: $\mu = 15000$

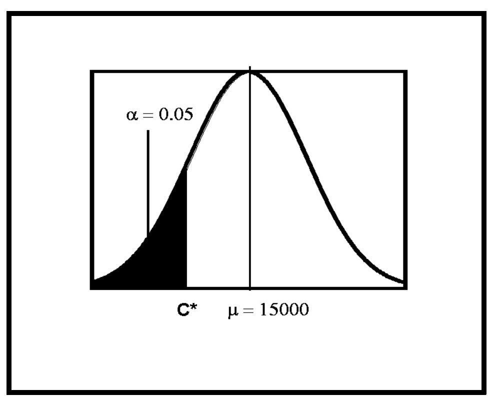
H_a: $\mu < 15000$

➤**(b)**.
Case 1: Type I error

Step 1: $\mu = 15{,}000$

$$c^* = \mu + z\sigma_{\bar{x}} = 1500 + z\frac{\sigma}{\sqrt{N}} = 1500 + z\frac{700}{\sqrt{N}}$$

Step 2: $P\{\overline{X} < c^* < 1500\} = 0.05$



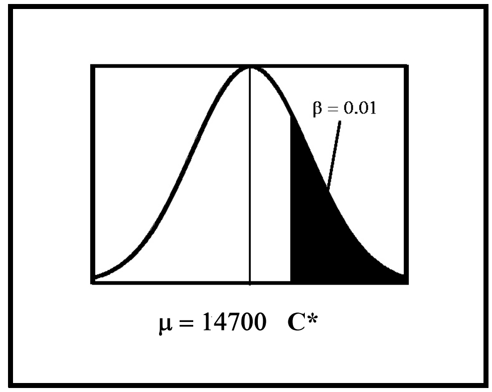
From the standard normal distribution table, we look-up z for $0.5 - 0.05 = 0.45$.

Therefore,

$z = -1.64$

$$c^* = 15000 + z\frac{700}{\sqrt{N}} = 15000 - (1.64)\frac{700}{\sqrt{N}} = 15000 - \frac{1148}{\sqrt{N}}$$

$c^* = 15000 - \dfrac{1148}{\sqrt{N}}$, equation 1



Case 2: Type II error

Step 1: $\mu = 14{,}700$

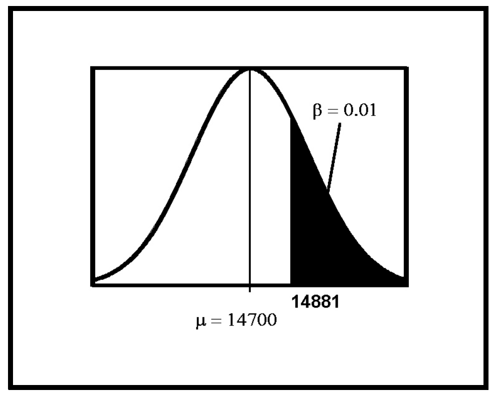
$c^* = \mu + z\sigma_{\bar{x}} = 14700 + z\dfrac{\sigma}{\sqrt{N}} = 14700 + z\dfrac{700}{\sqrt{N}}$

Step 2: $P\{14700 < c^* < \bar{X}\} = 0.01$

From the standard normal distribution table, we look-up z for 0.5 - 0.01 = 0.49: z = 2.33 .

$$c^* = 14700 + z\frac{700}{\sqrt{N}} = 14700 + (2.33)\frac{700}{\sqrt{N}} = 14700 + \frac{1631}{\sqrt{N}}$$

$c^* = 14700 + \dfrac{1631}{\sqrt{N}}$, equation 2



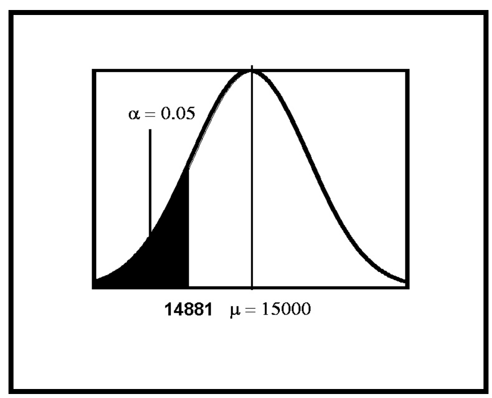
Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$$15000 - \frac{1148}{\sqrt{N}} = 14700 + \frac{1631}{\sqrt{N}}$$

$$300 = \frac{2779}{\sqrt{N}}$$

$$\sqrt{N} = \frac{2779}{300} \approx 9$$

N = 81



From equation 2,

$$c^* = 14700 + \frac{1631}{\sqrt{N}} = 14700 + \frac{1631}{9} \approx 14{,}881$$

➤**(c)**.

Decision Rule: Take a sample of size 81. If $\bar{X} < 14{,}881$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

Solved Problems

34.1 - Solved Problem 1: A U.S. Department of Agriculture study showed that over the last 50 years, cattle ranchers in southern Texas produced on average 25,200 head of cattle per year. As an attempt to increase this yield, a sample of N cattle ranches decided to feed their cattle a new hybrid of corn. After one year, the average number of cattle produced was 25,524. Assuming a standard deviation of 2,100 cattle,

(a). State H_o and H_a.

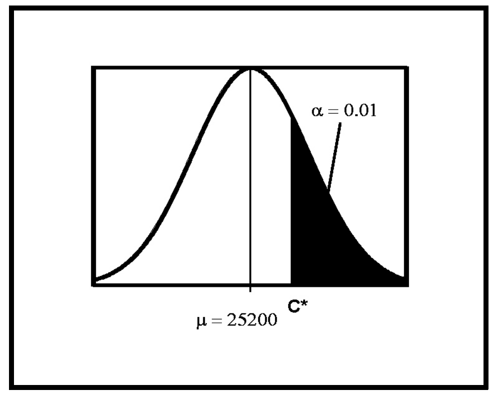
(b). For the following decision rule,

D.R.: Take a sample of size N. If $\overline{X} > c^*$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a*.*

Find N and c* so that $\alpha = 0.01$ and $\beta = 0.05$ when $\mu = 25{,}500$.

(c). Restate the decision rule.

Solutions:



➤**(a)**.
H_o: $\mu = 25{,}200$

H_a: $\mu > 25{,}200$

➤**(b)**.
Case 1: Type I error

Step 1: $\mu = 25{,}200$

$$c^* = \mu + z\sigma_{\bar{x}} = 25200 + z\frac{\sigma}{\sqrt{N}} = 25200 + z\frac{2100}{\sqrt{N}}$$

Step 2: $P\{25{,}200 < c^* < \overline{X}\} = 0.01$

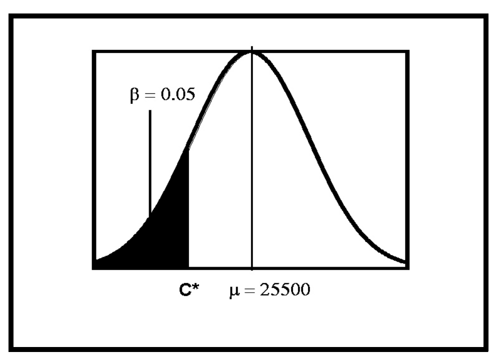
From the standard normal distribution table, we look-up z for
0.5 - 0.01 = .49 and find z = 2.33.

$$c^* = 25200 + z\frac{2100}{\sqrt{N}} = 25200 + (2.33)\frac{2100}{\sqrt{N}} = 25200 + \frac{4893}{\sqrt{N}}$$

$$c^* = 25200 + \frac{4893}{\sqrt{N}}, \text{ equation 1.}$$

Case 2: Type II error

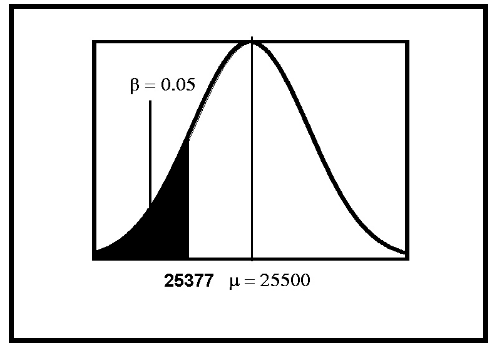
Step 1: $\mu = 25{,}500$



$$c^* = \mu + z\sigma_{\bar{x}} = 25500 + z\frac{\sigma}{\sqrt{N}} = 25500 + z\frac{2100}{\sqrt{N}}$$

Step 2: $P\{\overline{X} < c^* < 25{,}500\} = 0.05$

From the standard normal distribution table, we look-up z for 0.5 - 0.05 = .45: z = -1.64.



$$c^* = 25500 + z\frac{2100}{\sqrt{N}} = 25500 - (1.64)\frac{2100}{\sqrt{N}} = 25500 - \frac{3444}{\sqrt{N}}$$

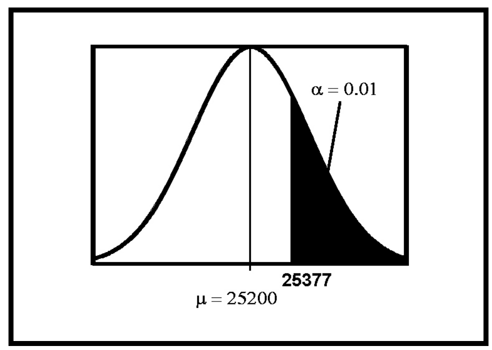
$$c^* = 25500 - \frac{3444}{\sqrt{N}}, \text{ equation 2}$$

Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

$$25200 + \frac{4893}{\sqrt{N}} = 25500 - \frac{3444}{\sqrt{N}}$$

$$300 = \frac{8337}{\sqrt{N}}$$

$$\sqrt{N} = \frac{8337}{300} \approx 28$$



N = 784

From equation 2,

$$c^* = 25500 - \frac{3444}{\sqrt{N}} = 25500 - \frac{3444}{28} = 25{,}377$$

➤**(c)**.
D.R.: Take a sample of size 784. If $\overline{X} > 25{,}377$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

34.1 - Solved Problem 2: The Sally Stone Speed Reading System claims that a person using their system, after six weeks, will be able to read at least 1,200 words a minute. To test this claim, N graduating students of this program were tested for their speed in reading. Assume a standard deviation of 90 words a minute.

(a). State H_o and H_a.

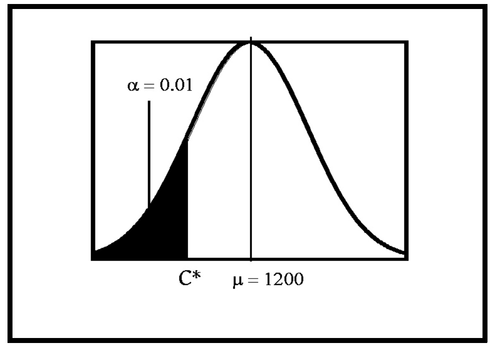
(b). For the following decision rule,

D.R.: Take a sample of size N. If $\overline{X} < c^*$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

Find N and c* so that $\alpha = 0.01$ and $\beta = 0.01$ when $\mu = 1{,}150$.

(c). Restate the decision rule.

Solutions:



➤**(a)**.
H_o: $\mu = 1200$

H_a: $\mu < 1200$

➤ **(b)**.
Case 1: Type I error.

Step 1: $\mu = 1{,}200$

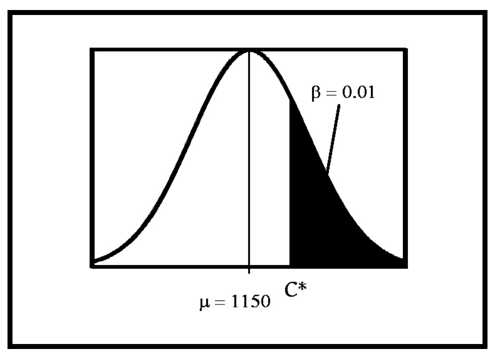
$$c^* = \mu + z\sigma_{\bar{x}} = 1200 + z\frac{\sigma}{\sqrt{N}} = 1200 + z\frac{90}{\sqrt{N}}$$

Step 2: $P\{\bar{X} < c^* < 1200\} = 0.01$

From the standard normal distribution table, we look-up z for 0.5 - 0.01 = 0.49 and we find z = -2.33 .

$$c^* = 1200 + z\frac{90}{\sqrt{N}} = 1200 - (2.33)\frac{90}{\sqrt{N}} = 1200 - \frac{209.7}{\sqrt{N}}$$

$$c^* = 1200 - \frac{209.7}{\sqrt{N}}, \text{ equation 1}$$



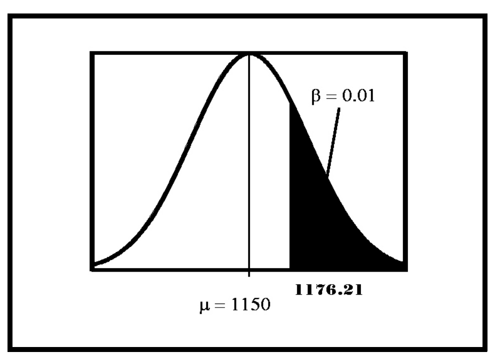
Case 2: Type II error

Step 1: $\mu = 1{,}150$

$$c^* = \mu + z\sigma_{\bar{x}} = 1500 + z\frac{\sigma}{\sqrt{N}} = 1500 + z\frac{90}{\sqrt{N}}$$

Step 2: $P\{1000 < c^* < \bar{X}\} = 0.01$

From the standard normal distribution table, we look-up z for 0.5 - 0.1 = 0.49:
z = 2.33 .



$$c^* = 1150 + z\frac{90}{\sqrt{N}} = 1150 + (2.33)\frac{90}{\sqrt{N}} = 1150 + \frac{290.7}{\sqrt{N}}$$

$$c^* = 1150 + \frac{209.7}{\sqrt{N}}, \text{ equation 2}$$

Since the c* and N are the same for equation 1 and equation 2, we set the two equations equal and solve first for N:

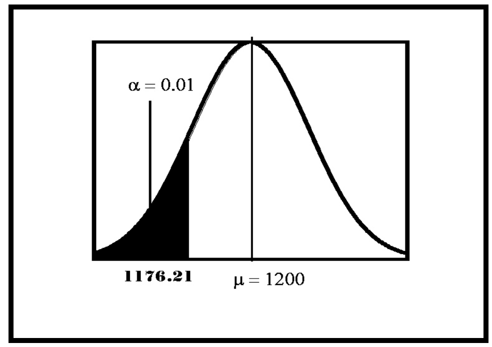
$$1200 - \frac{209.7}{\sqrt{N}} = 1150 - \frac{290.7}{\sqrt{N}}$$

$$\sqrt{N} = \frac{419.4}{50} \approx 8$$

N = 64

From equation 2,

$$c^* = 1150 + \frac{209.7}{8} \approx 1176.21$$



➤ **(c)**.
D.R.: Take a sample of size 64. If $\overline{X} < 1176.21$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

Unsolved Problem with Answers

34.1 - Problem 1: A certain manufacturing process has been used in the automobile industry to produce a part for the transmission system. This process, on average, takes 5.6 minutes per transmission system. The manufacturer of a new laser machine claims that their machine will decrease the average production time. Using this new machine, a time study was taken to determine the average time to produce N transmissions. This study resulted in an average time of 5.1 minutes per transmission system with a standard deviation of 0.45 minutes.

(a). State H_o and H_a.

(b). For the following decision rule,

D.R.: If $\overline{X} < c^*$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

Find N and c* so that $\alpha = 0.02$ and $\beta = 0.05$ when $\mu = 5.4$.

(c). Restate the decision rule.

Answers:

➤**(a)**.
H_o: $\mu = 5.6$

H_a: $\mu < 5.6$

➤**(b)**.
N = 69,

c* = 5.49

➤**(c)**.
D.R.: Take a sample of 64 transmissions. If $\overline{X} < 5.39$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

⥣ *Refer back to* **34.1 - Example 1 & 34.1 - Solved Problem 1**.

34.1 - Problem 2: The union claims that the average worker at their Oakland plant earns no more than $8.90 an hour. To test this claim, a sample of N workers is taken. Assuming a standard deviation 0f $1.00 .

(a). State H_o and H_a.

(b). For the following decision rule,

D.R.: If $\overline{X} > c^*$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

Find N and c* so that $\alpha = 0.10$ and $\beta = 0.05$ when $\mu = \$9.00$.

(c). Restate the decision rule.

Answers:

➤ **(a)**.
H_o: $\mu = \$8.90$

H_a: $\mu > \$8.90$

➤ **(b)**.
$N \approx 853$

$c^* = \$8.94$

➤ **(c)**.
D.R.: If $\overline{X} > \$8.94$*, then reject* H_o *and accept* H_a*; otherwise, reject* H_a.

⥣ *Refer back to* **34.1 - Example 2 & 34.1 - Solved Problem 2**.

34.2 - Creating decision rules from α and β for two-sided tests

34.2 - Example 1: At a book publisher's convention, an author of American history text books claimed that the average American history text book contains $\mu = 850$ pages. Assume you wish to test his claim by taking a random sample of N American history texts. Assume a standard deviation of $\sigma = 80$ pages.

(a). State H_o and H_a.

(b). For the following decision rule,

D.R.: If 850 - c ≤ $\overline{X}$ ≤ 850 + c*, then reject H_a; otherwise, reject H_0 and accept H_a.*

Find N and c* so that $\alpha = 0.05$ and $\beta = 0.05$ when $\mu = 870$ and $P\{\overline{X} < 850\} = \frac{\beta}{2} = 0.025$.

(c). Restate the decision rule.

Solutions:

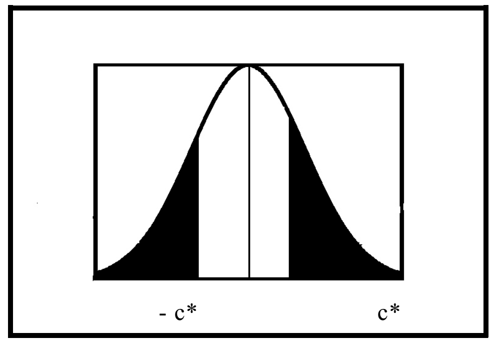
➤ **(a)**.

H_o: $\mu = 850$

$\mu \neq$

➤ **(b)**.

<u>Case 1: Type I error</u>



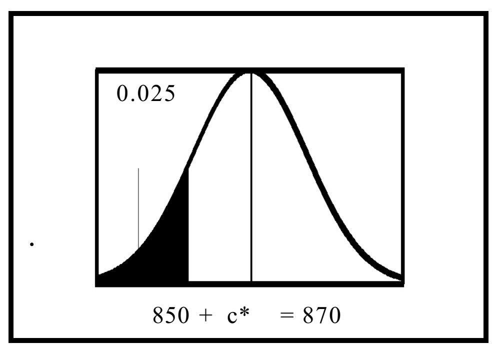
$\mu = 850$

$$c^* = z\sigma_{\bar{x}} = z\frac{\sigma}{\sqrt{N}} = z\frac{80}{\sqrt{N}}$$

From the standard normal distribution table, we look-up z for 0.5 - 0.025 = .475 and we find z = 1.96 .

$$c^* = z\frac{\sigma}{\sqrt{N}} = (1.96)\frac{80}{\sqrt{N}} = \frac{156.8}{\sqrt{N}}$$, equation 1

<u>Case 2: Type II error</u>



$\mu = 870$

Since we require $P\{\overline{X} < 850\} = \frac{\beta}{2} = 0.025$,

$$850 + c^* = 870 + z\frac{\sigma}{\sqrt{N}} = 870 + z\frac{80}{\sqrt{N}}$$

$$c^* = 20 + z\frac{80}{\sqrt{N}}$$

For $850 + c^* < \overline{X} < 870$, we have $P\{850 + c^* < \overline{X} < 870\} = 0.45$, $z = -1.64$

$$c^* = 20 + (-1.64)\frac{80}{\sqrt{N}} = 20 - \frac{131.2}{\sqrt{N}}, \text{ equation 2}$$

Since equation 1 and equation 2 are equal:

$$\frac{156.8}{\sqrt{N}} = 20 - \frac{131.2}{\sqrt{N}}$$

$$\frac{288}{\sqrt{N}} = 20$$

$$\sqrt{N} = \frac{288}{20}$$

$N \approx 207$

$$c^* = z\frac{\sigma}{\sqrt{N}} = (1.96)\frac{80}{\sqrt{N}} = \frac{156.8}{\left(\frac{288}{20}\right)} \approx 10.89$$

➤ **(c)**.
D.R.: Take a sample of size N = 207. If 850 -10.89 ≤ $\overline{X}$ *≤ 850 + 10.89, then reject* H_a*; otherwise, reject* H_0 *and accept* H_a*.*

Solved Problem

34.2 - Solved Problem 1: A large national corporation's past records show that their salespersons travel on average μ = 1,350 miles. They hire a statistician to determine if, during the past year, there has been a significant change in their travel mileage. A sample of N salespersons' travel records was taken. Assuming the standard deviation is σ = 150 miles.

(a). State H_o and H_a.

(b.) Assume μ = 1,250. For the following decision rule:

D.R.: If 1,350 - c ≤* $\overline{X}$ *≤ 1,350 + c*, then reject* H_a*; otherwise reject* H_o *and accept* H_a*.*

Find N and c* so that α = 0.05 and β = 0.02 when μ = 1,250 and

$$P\{\overline{X} > 1350\} = \frac{\beta}{2} = 0.01 .$$

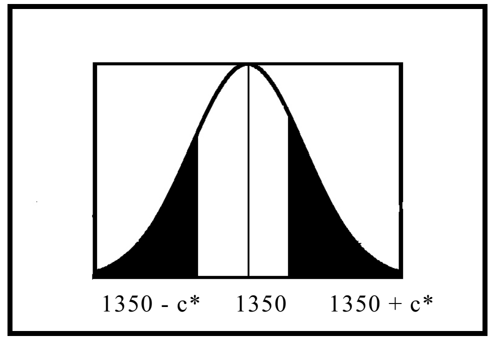
(c). Restate the decision rule.

Solutions:

➤ **(a)**.

H_o: $\mu = 1350$

H_a: $\mu \neq 1350$



➤ **(b)**.

Case 1: Type I error

Step 1: $\mu = 1350$

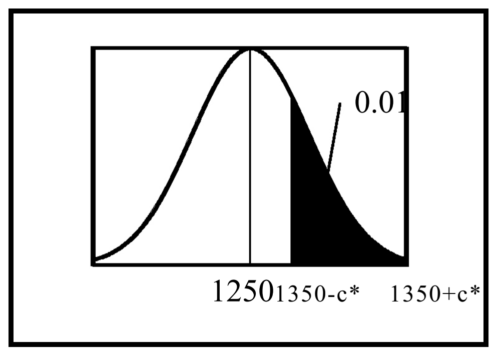
$$c^* = z\sigma_{\bar{x}} = z\frac{\sigma}{\sqrt{N}} = z\frac{150}{\sqrt{N}}$$

Step 2: From the standard normal distribution table, we look-up z for 0.5 - 0.025 = .475 and we find z = 1.96 .

$$c^* = z\frac{\sigma}{\sqrt{N}} = (1.96)\frac{150}{\sqrt{N}} = \frac{294}{\sqrt{N}}$$, equation 1

Case 2: Type II error

$\mu = 1250$



Since we require $P\{X > 1350\} = \frac{\beta}{2} = 0.01$,

$$1350 - c^* = 1250 + z\frac{\sigma}{\sqrt{N}} = 1250 + z\frac{150}{\sqrt{N}}$$

$$c^* = 100 - z\frac{150}{\sqrt{N}}$$

For $1250 < \overline{X} < 1350 - c^*$, we have $P\{1250 < \overline{X} < 1350 - c^*\} = 0.48$, $z = 2.05$

$$c^* = 100 - (2.05)\frac{150}{\sqrt{N}} = 100 - \frac{307.5}{\sqrt{N}}$$, equation 2

Since equation 1 and equation 2 are equal

$$\frac{294}{\sqrt{N}} = 100 - \frac{307.5}{\sqrt{N}}$$

$$\frac{601}{\sqrt{N}} = 100$$

$$\sqrt{N} = \frac{601}{100}$$

$N \approx 36$

$$c^* = z\frac{\sigma}{\sqrt{N}} = (2.05)\frac{150}{\sqrt{N}} = \frac{156.8}{(\frac{601}{100})} \approx 26.1$$

➤ **(c)**.
D.R.: Take a sample of size N = 36. If $1350 - 26.1 \le \overline{X} \le 1350 + 26.1$, *then reject* H_a*; otherwise, reject* H_0 *and accept* H_a.

Unsolved Problem with Answers

34.2 - Problem 1: Over many years, records have shown that in a certain large airport the average number of pieces of passenger luggage that was handled by the airport was 25,500 per day. Not satisfied with this number, the directors decided to modify the system that the employees used in handling the passengers' luggage. After completion, a random sample of 100 days was taken to find out if there had been any significant changes in the amount of luggage handled. Assume a standard deviation of 2,000.

(a). State H_o and H_a.

(b.) Assume $\mu = 25{,}500$. For the following decision rule:

D.R.: If $25{,}500 - c^* \le \overline{X} \le 25{,}500 + c^*$*, then reject* H_a*; otherwise reject* H_o *and accept* H_a.

Find N and c* so that $\alpha = 0.10$ and $\beta = 0.01$ when $\mu = 27{,}700$ and $P\{X < 25{,}500\} = \frac{\beta}{2} = 0.005$.

(c). Restate the decision rule.

Answers:

➤**(a)**.
H_o: $\mu = 25{,}500$

H_a: $\mu \ne 25{,}500$

➤**(b)**.
$N \approx 380$

$c^* \approx 82.62$

➤**(c)**.
D.R.: Take a sample of size N = 380. If 25,500 -82.62 ≤ $\overline{X}$ ≤ 25,500 + 82.62, then reject H_a; otherwise, reject H_o and accept H_a.

⥣ *Refer back to* **34.2 - Example 1 & 34.2 - Solved Problem 1**.

Supplementary Problems

1**.** Mrs. Pillar is running for reelection to Congress. Her opponent is against free trade. She will favor free trade if 50% or more of her district is in favor of free trade. She hires a political analysis to take a poll of 200 voters from her district to find out the number in favor of free trade. She decides on the following decision rule for supporting or rejecting free trade in her campaign:

D.R.: If at least 95 of the voters say they are in favor of free trade, she will state in her election advertisements that she supports free trade. However, if less than 95 say they are in favor of free trade, she will state in her election advertisements that she does not support free trade.

a. State H_o and H_a

b. Find the probability of a Type I error α.

c. If the true proportion in her district that support free trade is only 40%, find the probability that she will commit a Type II error β.

d. Modify the decision rule so that α = 0.05 when at least 50% of the voters support free trade and β = 0.01 when 40% of the voters support free trade.

2. A Federal agency believes that a national bank discriminates against a certain minority group when approving loans. This group constitutes 17% of the population. The agency decides to check 100 loans at random for evidence of discrimination. They use the following decision rule:

D.R.: If 16 or less of these loans are issued to members of this minority group, then the agency will conclude that the bank discriminates at this group; otherwise the agency will reserve judgement.

a. State H_o and H_a

b. Find the probability of a Type I error α.

c. If the true proportion of loans given to this group is 12%, find the Type II error β.

d. Modify the decision rule so that α = 0.02 when at least 17% of the loans are approved for this group and β = 0.05 when 12% of the loans are approved for this group.